Numerical Integration

If a function is so complicated and cannot be integrated, or, function value given in the form of table, then we use numerical Integration.

$$I = \int_a^b f(x)\,dx$$

This method gives an approximate value of this definite integral. ~~If the~~ So, if points are given or, if complicated $f(x)$ is given then we get the points first from the given function. (should be equidistant), then we use a suitable polynomial which is integratable is assigned through that points. And this integration process is called quadrature.

Say $f(x)$ is known for $(n+1)$ points.

they are $(x_0, y_0), (x_1, y_1), (x_2, y_2) \ldots, (x_n, y_n)$.

Here interpolating polynomial is $P(x)$

$$\therefore \int_a^b f(x)\,dx \simeq \int_a^b P(x)\,dx = I$$

then $\int_a^b f(x)\,dx - \int_a^b P(x)\,dx$, is called the error.

General quadrature formula (Gauss)

Ans

$$I = \int_a^b f(x)\,dx,$$

say ~~$a = x_0, b = x$~~

$x_0 = a, \; x_n = b,$

$x_1 = a + h, \quad x_2 = a + 2h,$

$\;\;\; = x_0 + h, \quad\; = x_0 + 2h$

$\Big\}\; x_n = x_0 + nh$

$\therefore \; b - a = x_n - x_0 = nh.$

$$\therefore I = \int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$$

Let $u = \frac{x - x_0}{h}$

$\therefore x = x_0 + hu$

$\therefore dx = h \cdot du$.

and if $x = x_0$, $u = 0$

if $x = x_n$, then $u = n$ $\left[\frac{x_n - x_0}{h} = \frac{nh}{h} = n\right]$

So,
$$I = \int_0^n f(x_0 + hu) \cdot h \cdot du$$

$$= h \int_0^n f(x_0 + hu) \cdot du$$

$$= h \cdot \int_0^n E^u f(x_0)\, du$$

$$= h \cdot \int_0^n (1+\Delta)^u f(x_0)\, du.$$

$$= h \cdot \int_0^n \left\{1 + u\Delta + \frac{u(u-1)}{2!}\Delta^2 + \frac{u(u-1)(u-2)}{3!}\Delta^3 + \cdots \right\} f(x_0) \cdot du.$$

$$= h \cdot \int_0^n \left[f(x_0) + u \cdot \Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \cdots \right] du$$

$$= h \int_0^n \left[y_0 + u \cdot \Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \right] du$$

$$= h\left[\int_0^n y_0 \cdot du + \int_0^n u \cdot \Delta y_0 \cdot du + \int_0^n \frac{1}{2}(u^2 - u) \cdot \Delta^2 y_0\, du + \int_0^n \frac{1}{6}(u^3 - 3u^2 + 2u) \cdot \Delta^3 y_0\, du + \cdots \right]$$

$$= h\left[y_0 \cdot \int_0^n du + \Delta y_0 \cdot \int_0^n u\, du + \frac{\Delta^2 y_0}{2}\int_0^n (u^2 - u)\, du + \cdots \right]$$

$$= h\left\{y_0 \cdot [u]_0^n + \Delta y_0 \cdot \left[\frac{u^2}{2}\right]_0^n + \frac{\Delta^2 y_0}{2} \cdot \left[\frac{u^3}{3} - \frac{u^2}{2}\right]_0^n + \frac{\Delta^3 y_0}{6}\left[\frac{u^4}{4} - u^3 + u^2\right]_0^n + \cdots \right\}$$

$$= h\left\{ y_0(n-0) + \Delta y_0\left(\frac{n^2}{2} - \frac{0}{2}\right) + \frac{\Delta^2 y_0}{2}\left(\frac{n^3}{3} - \frac{n^2}{2} - \frac{0}{3} + \frac{0}{2}\right) \cdots \right\}$$

~~$= h\{ny_0 + \Delta y_0($~~

$$= h\left\{ ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{2n^3 - 3n^2}{12}\Delta^2 y_0 + \frac{n^4 - 4n^3 + 4n^2}{24}\Delta^3 y_0 + \cdots \right\}$$

This equation is called gauss legendre Quadrature formula. Now we can get all quadrature formula's from this Gauss quadrature formula.

There are three quadrature formula, are there based on Gauss quadrature formula.

1) Trapezoidal Rule
2) Simpson's one third rule
3) Weddle's rule.

Trapezoidal Rule

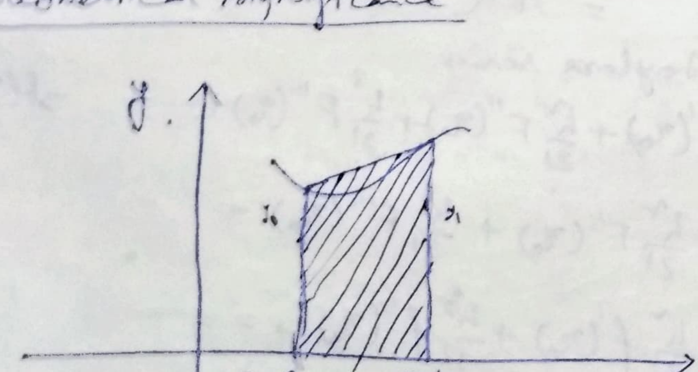
It is two point quadrature formula, that is

$n+1 = 2 \quad \therefore \quad n = 2-1 = 1$

$\therefore n = 1$

that is for a Integration $\int_a^b f(x)\,dx$. we use only two values $(x_0 = a, y_0)$ and $(x_1 = b, y_1)$.

Geometrical significance



So the integration means the area of the trapezium shown.

So the integration value

$$I_T = \frac{(x_1 - x_0)}{2}\cdot(y_0 + y_1)$$

is the area of trapezium's formula is $\frac{h}{2}(a+b)$.
where h is height and a, b are two sides.

here
$$I_T = \frac{(x_1 - x_0)}{2} \cdot \{y_0 + y_1\}$$
$$= \frac{(b-a)}{2} \cdot \{f(a) + f(b)\}$$
$$\therefore I_T = \frac{h}{2}\{f(a) + f(b)\} \quad \text{when } h = b-a.$$

now error committed by this formula is given by

$$E_T = \int_a^b f(x)\,dx - I_T$$

(Error in trapezoidal method)
$$= \int_{x_0}^{x_0+h} f(x)\,dx - \frac{h}{2}\{f(x_0) + f(x_1)\}.$$

Let us assume $f(x)$ is finite and continuous in the interval $[x_0, x_1]$ and posses derivative ~~of~~ of all orders.

Let $\frac{d}{dx}F(x) = f(x)$

$\therefore \quad F'(x) = f(x).$

now $F(x) = \int f(x)\,dx$

$$\therefore \int_{x_0}^{x_0+h} f(x)\,dx = \{F(x)\}_{x_0}^{(x_0+h)}$$
$$= F(x_0+h) - F(x_0)$$

So by using taylors series,
$$= \cancel{F(x_0)} + hF'(x_0) + \frac{h^2}{2!}F''(x_0) + \frac{h^3}{3!}F'''(x_0) + \cdots - \cancel{F(x_0)}$$
$$= h.F'(x_0) + \frac{h^2}{2!}F''(x_0) + \frac{h^3}{3!}F'''(x_0) + \cdots$$
$$= h.f(x_0) + \frac{h^2}{2!}f'(x_0) + \frac{h^3}{3!}f''(x_0) + \cdots$$
$$= h.f(x_0) + \frac{h^2}{2}f'(x_0) + \frac{h^3}{6}f''(x_0) + \cdots$$

-- -- -- --

Again

$$I_T = \frac{h}{2}\left[f(x_0) + f(x_0+h) \right]$$

$$= \frac{h}{2}\left[f(x_0) + f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots \right]$$

$$= \frac{h}{2}\left[2f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{6} f'''(x_0) + \cdots \right]$$

$$= h f(x_0) + \frac{h^2}{2} f'(x_0) + \frac{h^3}{4} f''(x_0) + \frac{h^4}{12} f'''(x_0) + \cdots \quad (2)$$

So from (1) and (2) we have

(1) − (2).

Then

$$E_T = \left\{ h f(x_0) + \frac{h^2}{2} f'(x_0) + \frac{h^3}{6} f''(x_0) + \frac{h^4}{24} f'''(x_0) + \cdots \right\} - \left\{ h f(x_0) + \frac{h^2}{2} f'(x_0) + \frac{h^3}{4} f''(x_0) + \frac{h^4}{12} f'''(x_0) + \cdots \right\}$$

$$= \cancel{h f(x_0)} + \cancel{\frac{h^2}{2} f'(x_0)} + \frac{h^3}{6} f''(x_0) + \frac{h^4}{24} f'''(x_0) + \cdots - \cancel{h f(x_0)} - \cancel{\frac{h^2}{2} f'(x_0)} - \frac{h^3}{4} f''(x_0) - \frac{h^4}{12} f'''(x_0) - \cdots$$

$$= \left(\frac{h^3}{6} - \frac{h^3}{4}\right) f''(x_0) + \left(\frac{h^4}{24} - \frac{h^4}{12}\right) f'''(x_0) + \cdots$$

$$\cong \left(\frac{1}{6} - \frac{1}{4}\right) h^3 f''(x_0)$$ by neglecting higher order terms

$$= -\frac{h^3}{12} f''(x_0)$$

$$\therefore \quad E_T \cong -\frac{h^3}{12} f''(x_0).$$

Composite rule of Trapezoidal method.

If the range of integration $[a,b]$, is divided into n-equal sub-intervals, by the points

$(x_0,y_0), (x_1,y_1), (x_2,y_2) \cdots (x_n,y_n)$

where $x_n = x_0 + nh$ and $y_n = f(x_n)$, where $x_0 = a$, $x_n = b$.

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= \int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \int_{x_0+2h}^{x_0+3h} f(x)\,dx + \cdots + \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{h}{2}\left[f(x_0)+f(x_0+h)\right]+\frac{h}{2}\left[f(x_0+h)+f(x_0+2h)\right]+\frac{h}{2}\left[f(x_0+2h)+f(x_0+3h)\right]$$
$$+ \cdots + \frac{h}{2}\left[f(x_0+(n-1)h)+f(x_0+nh)\right]$$
$$= \frac{h}{2}\Big[f(x_0)+f(x_0+h)+f(x_0+h)+f(x_0+2h)+f(x_0+2h)+f(x_0+3h)$$
$$+ \cdots f(x_0+(n-1)h + f(x_0+(n-1)h)+f(x_0+nh)\Big]$$
$$= \frac{h}{2}\Big[\{f(x_0)+f(x_0+nh)\}+2\{f(x_0+h)+f(x_0+2h)$$
$$+ \cdots f(x_0+(n-1)h)\}\Big]$$
$$= \frac{h}{2}\Big[\{f(x_0)+f(x_n)\}+2\{f(x_1)+f(x_2)+\cdots+f(x_{n-1})\}\Big]$$
$$= \frac{h}{2}\Big[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\Big]$$
$$= \frac{h}{2}\times\Big[\text{Sum of first and last ordinates} + 2\times\text{sum of all other ordinates}\Big]$$
$$= I_T^C \quad [\text{Integration value by composite trapezoidal rule}]$$

Here error committed is simply multiplied by a factor of 'n', but interval length 'h' is reduced by n times.

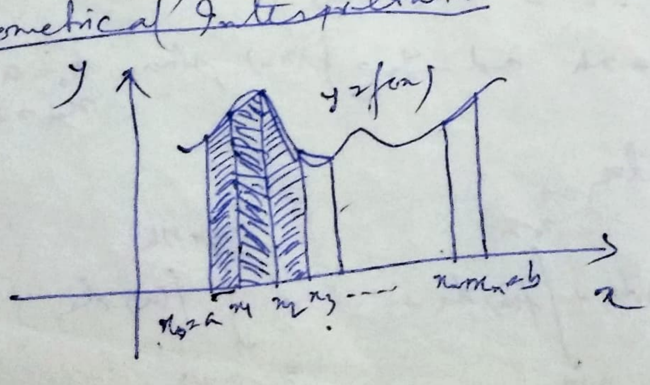
so, $E_T^C \cong -n.\frac{h^3}{12}f''(x_0)$

where $h = \frac{b-a}{n}$

previously $h = (b-a)$

So ultimately error will reduced.

Geometrical Interpretation



Simpson's One-third rule :-

It is a three point quadrature formula, that is here $n+1 = 3$.

$\therefore n = 3-1 = 2$

$\therefore n = 2$.

here only three functional points are

(x_0, y_0), (x_1, y_1), (x_2, y_2)

where $y_0 = f(x_0) = f(a)$, $y_1 = f(x_1) = f(x_0+h)$

$y_2 = f(x_2) = f(x_0+2h) = f(x_n) = f(b)$

now,
the Gauss-legendre quadrature formula is, given as.

$$I = h\left\{ ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{2n^3-3n^2}{12}\Delta^2 y_0 + \frac{n^4-4n^3+4n^2}{24}\Delta^3 y_0 + \cdots \cdots \right\}$$

[if $n=1$, then formula comes.

$$I_T = h\left\{ 1.y_0 + \frac{1^2}{2}.\Delta y_0 + \frac{2.1^3-3.1^2}{12}.\Delta^2 y_0 + \cdots \right\}$$

$$= h\left\{ y_0 + \frac{1}{2}.\Delta y_0 + \frac{2-3}{12}.0 + \cdots \right\}$$

$$= h\left\{ y_0 + \frac{1}{2}\Delta y_0 \right\}$$

$$= h\left\{ y_0 + \frac{1}{2}(y_1 - y_0) \right\}$$

$$= h\left\{ y_0 + \frac{1}{2}y_1 - \frac{1}{2}y_0 \right\}$$

$$= h\left\{ \frac{1}{2}y_0 + \frac{1}{2}y_1 \right\}$$

$\therefore I_T = \frac{h}{2}\{ y_0 + y_1 \}$ formula comes for trapezoidal rule]

now, if we put $n=2$ in above gauss-legendre quadrature formula then;

$$I_S = h\left\{ 2.y_0 + \frac{2^2}{2}\Delta y_0 + \frac{2.2^3-3.2^2}{12}.\Delta^2 y_0 + \frac{2^4-4.2^3+4.2^2}{24}\Delta^3 y_0 + \cdots \right\}$$

$$= h\left\{ 2y_0 + \frac{4}{2}\Delta y_0 + \frac{2.8-3.4}{12}\Delta^2 y_0 + \frac{16-4.8+4.4}{24}\Delta^3 y_0 + \cdots \right\}$$

$$= h\left\{ 2y_0 + 2.\Delta y_0 + \frac{16-12}{12}\Delta^2 y_0 + \frac{16-32+16}{24}.0 + \cdots \right\}$$

$$\therefore I_s = h\left\{2y_0 + 2\Delta y_0 + \frac{4}{12}\cdot\Delta^2 y_0\right\}$$

$$= h\left\{2y_0 + 2(y_1 - y_0) + \frac{1}{3}(\Delta y_1 - \Delta y_0)\right\}$$

$$= h\left\{\cancel{2y_0} + 2y_1 - \cancel{2y_0} + \frac{1}{3}(\Delta y_1) - \frac{1}{3}(\Delta y_0)\right\}$$

$$= h\left\{2y_1 + \frac{1}{3}(\Delta y_1) - \frac{1}{3}(\Delta y_0)\right\}$$

$$= \frac{h}{3}\left\{6y_1 + (y_2 - y_1) - (y_1 - y_0)\right\}$$

$$= \frac{h}{3}\left\{6y_1 + y_2 - y_1 - y_1 + y_0\right\}$$

$$= \frac{h}{3}\left\{y_0 + 4y_1 + y_2\right\}$$

$$\therefore I_s = h/3\left\{y_0 + 4y_1 + y_2\right\}$$

Now, Error committed in Simpson's $1/3$ rule is

$$E_s = \int_a^b f(x)\,dx - I_s$$

$$= \int_{x_0}^{x_0+2h} f(x)\,dx - \frac{h}{3}\left\{y_0 + 4y_1 + y_2\right\}$$

as, it is 3 point quadrature formula.

therefore $x_0 = a$, and $x_2 = x_0 + 2h = b$

and x_1 be the intermediate point.

$$\therefore E_s = \int_{x_0}^{x_0+2h} f(x)\,dx - \frac{h}{3}\left\{f(x_0) + 4f(x_1) + f(x_2)\right\}$$

Let $F'(x) = f(x)$ and, $f(x)$ is a continuous and derivable in all order, say

$$\therefore \int_a^b f(x)\,dx = \left[F(x)\right]_a^b$$

$$\therefore \int_{x_0}^{x_0+2h} f(x)\,dx = \left[F(x)\right]_{x_0}^{x_0+2h}$$

$$= F(x_0 + 2h) - F(x_0)$$

By using taylor's series expansion formula we h

Thus
$$\int_{x_0}^{x_0+2h} f(x)\,dx = \cancel{F(x_0)} + 2h\,F'(x_0) + \frac{(2h)^2}{2!}F''(x_0) + \frac{(2h)^3}{3!}F'''(x_0) + \frac{(2h)^4}{4!}F^{iv}(x_0) \cancel{- F(x_0)} + \frac{(2h)^5}{5!}F^{v}(x_0) + \cdots$$

~~by neglecting ...~~

$$\therefore \quad = 2h\,F'(x_0) + \frac{2 \cancel{4} h^2}{\cancel{2}}F''(x_0) + \frac{\overset{4}{\cancel{8}} h^3}{3\,\cancel{6}}F'''(x_0) + \frac{\overset{2}{\cancel{16}} h^4}{3\,\cancel{24}}F^{iv}(x_0) + \frac{\overset{4}{\cancel{32}} h^5}{\underset{15}{\cancel{120}}}F^{v}(x_0) + \cdots$$

$$= 2h\,F'(x_0) + 2h^2 F''(x_0) + \frac{4}{3}h^3 F'''(x_0) + \frac{2}{3}h^4 F^{iv}(x_0) + \frac{4}{15}h^5 F^{v}(x_0) \cdots$$

by putting $F'(x_0) = f(x)$

$$\therefore \int_{x_0}^{x_0+2h} f(x)\,dx = 2h f(x_0) + 2h^2 f'(x_0) + \frac{4}{3}h^3 f''(x_0) + h^4 f'''(x_0) + \frac{4}{15}h^5 f^{iv}(x_0) + \cdots \quad (1)$$

Again
$$I_s = \frac{h}{3}\left[f(x_0) + 4.f(x_1) + f(x_2)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4.f(x_0+h) + f(x_0+2h)\right]$$

$$= \frac{h}{3}\Big[f(x_0) + 4\Big\{f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \frac{h^4}{4!}f^{iv}(x_0) + \cdots\Big\}$$
$$+ \Big\{f(x_0) + 2hf'(x_0) + \frac{(2h)^2}{2!}f''(x_0) + \frac{(2h)^3}{3!}f'''(x_0) + \cdots\Big\}\Big]$$

$$= \frac{h}{3}\Big[f(x_0) + 4f(x_0) + 4hf'(x_0) + 2h^2 f''(x_0) + \frac{2h^3}{3}f'''(x_0) + \frac{h^4}{6}f^{iv}(x_0) + \frac{h^5}{30}f^{v}(x_0) + f(x_0) + 2hf'(x_0) + 2h^2 f''(x_0)$$
$$+ \frac{4}{3}h^3 f'''(x_0) + \frac{2}{3}h^4 f^{iv}(x_0) + \frac{4}{15}h^5 f^{v}(x_0) + \cdots\Big]$$

$$= \frac{h}{3}\Big[6f(x_0) + 6hf'(x_0) + 4h^2 f''(x_0) + 2h^3 f'''(x_0) + \frac{5}{6}h^4 f^{iv}(x_0) + \frac{3}{10}h^5 f^{v}(x_0) + \cdots\Big]$$

$$= 2hf(x_0) + 2h^2 f'(x_0) + \frac{4}{3}h^3 f''(x_0) + \frac{2}{3}h^4 f'''(x_0) + \frac{5}{18}h^5 f^{iv}(x_0) + \frac{1}{10}\cdot h^6 f^{v}(x_0) + \cdots \quad (2)$$

So by getting ① − ②, we have

$$E_S = \int_{x_0}^{x_0+2h} f(x)\,dx - I_S.$$

$$= \cancel{2hf(x_0)} + \cancel{2h^2 f'(x_0)} + \cancel{\frac{4}{3}h^3 f''(x_0)} + \cancel{\frac{2}{3}h^4 f'''(x_0)} + \frac{4}{15}h^5 f^{iv}(x_0) + \cdots$$

$$- \cancel{2hf(x_0)} - \cancel{2h^2 f'(x_0)} - \cancel{\frac{4}{3}h^3 f''(x_0)} - \cancel{\frac{2}{3}h^4 f'''(x_0)}$$

$$- \frac{5}{18}h^5 f^{iv}(x_0) - \frac{1}{10}h^6 f^{v}(x_0) - \cdots$$

$$\cong \left(\frac{4}{15} - \frac{5}{18}\right) h^5 f^{iv}(x_0)$$ [by neglecting sixth and higher order terms]

$$= \left(\frac{24-25}{90}\right) h^5 f^{iv}(x_0)$$

$$\therefore E_S = -\frac{h^5}{90} f^{iv}(x_0)$$

Composite Simpson's $\frac{1}{3}$ rule

~~I~~ let we ~~do~~ divide the total interval into $n = 2m$ interval and there will be $2m+1$ points.

then, $a = x_0$, and other points are x_0+h, x_0+2h, x_0+3h, ~~and~~ x_0+4h, x_0+5h + ⋯ + $x_0+(2m-2)h$ + $x_0+(2m-1)h$ + $x_0+2mh = b = x_n$

then we apply Simpson's $\frac{1}{3}$ rule for each m subintervals

i.e, $[x_0, x_0+2h]$, $[x_0+2h, x_0+4h]$, $[x_0+4h, x_0+6h]$; ------ $[x_0+2(m-2)h, x_0+2mh]$.

$$\therefore I_S^c = \int_a^b f(x)\,dx = \int_{x_0}^{x_0+2mh} f(x)\,dx$$

$$= \int_{x_0}^{x_0+2h} f(x)\,dx + \int_{x_0+2h}^{x_0+4h} f(x)\,dx + \cdots + \int_{x_0+2(m-1)h}^{x_0+2mh} f(x)\,dx$$

$$\therefore I_s^c = \frac{h}{3}\left[f(x_0) + 4f(x_0+h) + f(x_0+2h)\right] + \frac{h}{3}\left[f(x_0+2h) + 4f(x_0+3h) + f(x_0+4h)\right]$$

$$+ \frac{h}{3}\left[f(x_0+4h) + 4f(x_0+5h) + f(x_0+6h)\right] + \cdots$$

$$+ \frac{h}{3}\left[f(x_0+(2m-2)h) + 4f(x_0+(2m-1)h) + f(x_0+2mh)\right]$$

$$= \frac{h}{3} \times \Big[\{f(x_0) + f(x_0+2mh)\} + 4\{f(x_0+h) + f(x_0+3h) + f(x_0+5h) + \cdots + f(x_0+(2m-1)h)\}$$

$$+ 2\{f(x_0+2h) + f(x_0+4h) + f(x_0+6h) + \cdots + f(x_0+(2m-2)h)\}\Big]$$

$$= \frac{h}{3} \times \Big[(y_0 + y_{2m}) + 4(y_1 + y_3 + y_5 + \cdots + y_{2m-1})$$

$$+ 2(y_2 + y_4 + y_6 + \cdots + y_{2m-2})\Big]$$

$= \frac{h}{3} \times$ [{Sum of first ad last term}

+ 4 × {sum of all other odd terms}

+ 2 × {sum of all other even terms}]

Error committed

$$E_s^c = \cancel{\frac{h^5}{90}} - \sum_{i=1}^{m} \frac{h_c^5}{90} f^{iv}(x_0)$$

$$= -\frac{m h_c^5}{90} \cdot f^{iv}(x_0) \quad \text{defined in } E_s$$

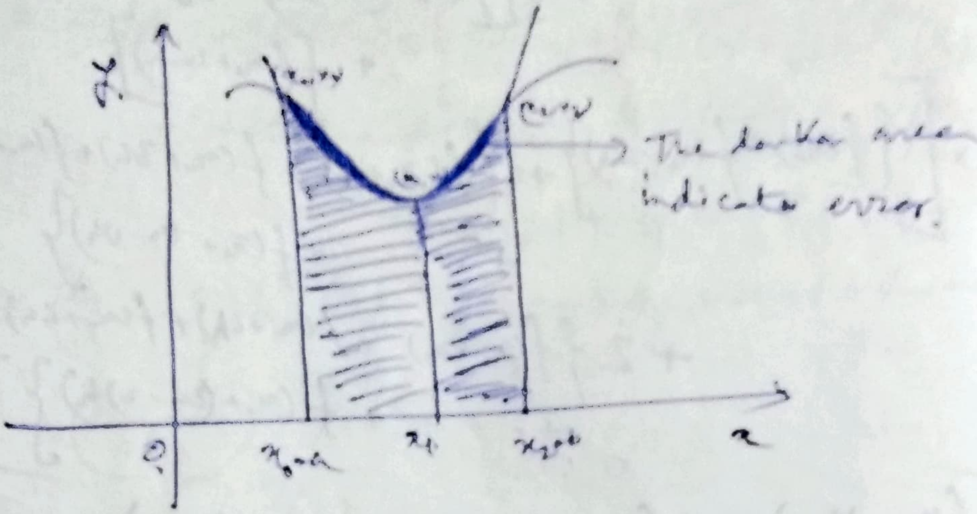
here $h_c = \frac{h}{2m}$ $\therefore h_c^5 = \frac{h^5}{32 m^5}$

$$\therefore E_s^c = -\frac{m \cdot h^5}{32 m^5 \cdot 90} f^{iv}(x_0)$$

$$= -\frac{h^5}{2880\, m^4} \cdot f^{iv}(x_0).$$

Geometrical Interpretation

Here the curve $y=f(x)$ is replaced by three point associated parabola



Algorithm for Composite Trapezoidal Rule-

1. Take input as limits, upperlimit and lowerlimit
   a = lower limit
   b = upper limit
2. take input no. of interval as 'n'
3. set $h = \frac{b-a}{n}$
4. for (i = 0 to n) incremented by 1,
5. set. $y_i = f(a + i.h)$.
6. endfor.
7. set sum = f(a)+f(b). ~ $y_0 + y_n$
8. for (i=1 to n-1) incremant by 1
9. set sum= sum + y_i*2
10. endfor
11. set sum = sum * (h/2).
12. Print sum. as result
13. exit.

Algorithm for Composite Simpsons $\frac{1}{3}$ rule :-

1. take input a = lower limit
2. take input b = upper limit
3. take input n = no. of interval
4. set $h = \frac{b-a}{n}$
5. for (i = 0 to n) incremented by 1.
6. set $y_i = f(a + ih)$
7. endfor.
8. set sum = f(a) + f(b) or $y_0 + y_n$
9. for (i = 1 to n-1) incremented by 2.
10. set sum = sum + 4 * y_i
11. end for.
12. for (i = 2 to n-2) incremented by 2
13. set sum = sum + 2 * y_i
14. endfor.
15. set sum = sum * (h/3)
16. Print sum as result.
17. exit.

Q1 Evaluate $\int_0^1 (4x - 3x^2)\, dx$, take 10 intervals

solve by ① Trapezoidal rule.

② Simpson's $\frac{1}{3}$ rule.

Also compute absolute and relative error

Q2 Calculate $\int_0^1 \frac{x}{1+x}\, dx$, take 6 intervals.

solve by ① Trapezoidal rule 0.305

② Simpson's $\frac{1}{3}$ rule 0.307

Also calculate relative and absolute error.

Ans 9.1

$\int_0^1 (4x-3x^2)\,dx$, taking 10 intervals

$\because y = f(x) = 4x - 3x^2$ and $n = 10$, $a = 0$, $b = 1$

therefore $h = \frac{b-a}{n} = \frac{1-0}{10} = \frac{1}{10} = 0.1$

$\therefore h = 0.1$

now

x_i	y_i
$x_0 = 0.0$	$y_0 = 0.00$
$x_1 = 0.1$	$y_1 = 0.37$
$x_2 = 0.2$	$y_2 = 0.68$
$x_3 = 0.3$	$y_3 = 0.93$
$x_4 = 0.4$	$y_4 = 1.12$
$x_5 = 0.5$	$y_5 = 1.25$
$x_6 = 0.6$	$y_6 = 1.32$
$x_7 = 0.7$	$y_7 = 1.33$
$x_8 = 0.8$	$y_8 = 1.28$
$x_9 = 0.9$	$y_9 = 1.17$
$x_{10} = 1.0$	$y_{10} = 1.00$

now, integration value using Trapezoidal rule :

$I_T^c = \frac{h}{2} \times [(\text{Sum of first + last term}) + 2 \times (\text{all other term})]$

$= \frac{h}{2} [(y_0 + y_{10}) + 2 \times (y_1 + y_2 + \cdots y_9)]$

$= \frac{0.1}{2} [(0.00 + 1.00) + 2(0.37 + 0.68 + 0.93 + 1.12 + 1.25 + 1.32 + 1.33 + 1.28 + 1.17)]$

$= \frac{0.1}{2} [1.00 + 2 \times 9.45]$

$= \frac{0.1}{2} [1.00 + 18.90]$

$= \frac{0.1}{2} [19.90] = \frac{0.1}{\cancel{2}} \times \cancel{19.90}^{9.95}$

$= 0.995$

the exact value is

$\int_0^1 (4x-3x^2)\,dx = \left[4 \cdot \frac{x^2}{2} - 3 \cdot \frac{x^3}{3}\right]_0^1 = \left[2x^2 - x^3\right]_0^1$

$= (2-1) - 0 = 1.00$

$\therefore \int_0^1 (4x-3x^2)\,dx = 1.000$

So absolute error in trapezoidal rule

is $= (1.000 - 0.995) = 0.005$

$\therefore$ Relative error $= \frac{0.005}{1.0} = 0.005$

Now by using Simpson's 1/3 rule
the integration value is

$$I_s^c = \frac{h}{3} \times \Big[(\text{sum of first and last term}) + 4(\text{all other odd terms}) + 2\times(\text{all other even terms})\Big]$$

$$= \frac{h}{3} \times \Big[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\Big]$$

$$= \frac{0.1}{3}\Big[(0.00 + 1.00) + 4(0.37 + 0.93 + 1.25 + 1.33 + 1.17) + 2(0.68 + 1.12 + 1.32 + 1.28)\Big]$$

$$= \frac{0.1}{3}\Big[1.00 + (4\times 5.05) + (2\times 4.40)\Big]$$

$$= \frac{0.1}{3}\Big[1.00 + 20.20 + 8.80\Big]$$

$$= \frac{0.1}{3}\Big[1.00 + 29.00\Big] = \frac{0.1}{\cancel{3}} \times \cancel{30.00}^{10.00} = 0.1\times 10.00 = 1.00.$$

$$\therefore I_s^c = 1.00$$

therefore Absolute error $= (1.000 - 1.000) = 0.00$.

and relative error $= \frac{0.00}{1.00} = 0.00$.

Romberg's Integration

This formula is a error correction formula basically using Trapezoidal rule, but other method like Simpson's 1/3 rule or weddle's rule can be used.

From Euler-Maclaurin formula we know that, Error computed in Trapezoidal rule is

$$\int_a^b f(x)\,dx - T(h) = a_2 h^2 + a_4 h^4 + a_6 h^6 + \cdots$$

where $T(h)$ indicates Integration value using Trapezoidal rule, taking h as a interval i.e $h = \frac{b-a}{n}$.
interval length

let $T(h, 0) = T(h)$
error level.

therefore

$$I = \int_a^b f(x)\,dx = T(h,0) + a_2 h^2 + a_4 h^4 + a_6 h^6 + \cdots \quad \text{--- (i)}$$

now, put $h/2$ in place of h.

then we have

$$I = \int_a^b f(x)\,dx = T(h/2, 0) + \frac{a_2}{4} h^2 + \frac{a_4}{16} h^4 + \frac{a_6 h^6}{64} + \cdots$$

by multiplying both side using 4 we have

$$4I = 4.T(h/2, 0) + a_2 h^2 + \frac{a_4}{4} h^4 + \frac{a_6}{16} h^6 + \cdots \quad \text{--- (ii)}$$

by doing (ii) − (i) we have

$$4I - I = 4.T(h/2, 0) - T(h,0) + \left(\frac{a_4}{4} - a_4\right) h^4 + \cdots$$

$$\text{or } 3I = 4.T(h/2, 0) - T(h,0) + \frac{3}{4} a_4 h^4 + \cdots$$

$$\therefore I = \frac{4.T(h/2,0) - T(h,0)}{3} + \frac{1}{4} a_4 h^4 + \cdots$$

$$\therefore I = \frac{4.T(h/2,0) - T(h,0)}{3} + b_4 h^4 + b_6 h^6 + \cdots$$

let $T(h/2, 1) = \dfrac{4\,T(h/2, 0) - T(h, 0)}{3}$, ← 1st level error correction

$\therefore I = T(h/2, 1) + b_4 h^4 + b_6 h^6 + \cdots \quad \text{(iii)}$

put, $h/2$ in place of h we have

$$I = T(h/4, 1) + \frac{b_4}{16} \cdot h^4 + \frac{b_6}{64} h^6 + \cdots$$

So by multiplying both side by 16 we have

$$16 I = 16 \cdot T(h/4, 1) + b_4 h^4 + \frac{b_6}{4} h^6 + \cdots \quad \text{(iv)}$$

by doing (iv) − (iii) we have

$$16I - I = 16\,T(h/4, 1) - T(h/2, 1) + \left(b_6 - \frac{b_6}{4}\right) h^6 + \cdots$$

So $I = \dfrac{16\,T(h/4, 1) - T(h/2, 1)}{15} + c_6 h^6 + \cdots$ ← Second level error correction

or $I = T(h/4, 2) + c_6 h^6 + \cdots \qquad \therefore T(h/4, 2) = \dfrac{16T(h/4, 1) - T(h/2, 1)}{16 - 1}$

Similarly we have

$$T(h/8, 3) = \frac{64\,T(h/8, 2) - T(h/4, 2)}{64 - 1}$$

So in general

$$T(h/2^i, j) = \frac{4^j\,T(h/2^i, j-1) - T(h/2^{i-1}, j-1)}{4^j - 1}$$

or if $R_{ij} = T(h/2^i, j)$ then

then $$R_{ij} = \frac{4^j R_{i, j-1} - R_{i-1, j-1}}{4^j - 1}$$

$\therefore I = R_{ij}$, ~~removes~~ corrects up to nth orders

called Romberg's integration formula.

j = levels of improvement
i = depth of division. $\quad \underline{j <= i}$