Numerical Solution of linear system of equations

The equations which have the form

$ax+by=c$, → are called linear equations.

but $ax^2+by=c$ → i.e other than that type of equations is called non-linear equations.

A linear equation with n variables has the form.

$a_1x_1+a_2x_2+a_3x_3+\cdots+a_nx_n=b$.

A set of equations is known as system of simultaneous equations or simply ~~set~~ system of equations.

A system of n linear equations is represented by.

$$
\begin{aligned}
a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n&=b_1\\
a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n&=b_2\\
\cdots\quad\cdots\quad\cdots&\\
a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n&=b_n.
\end{aligned}
$$

In matrix notation it can be written as.

$Ax=b$.

where A is coefficient matrix, X is the solution vector.

b is the R.H.S vector.

if $b_i=0$, the system is called homogenous otherwise the system is called non-homogenous.

Here

$$
b=\begin{bmatrix} b_1\\ b_2\\ b_3\\ \vdots\\ b_n\end{bmatrix},\quad
X=\begin{bmatrix} x_1\\ x_2\\ x_3\\ \vdots\\ x_n\end{bmatrix},\quad
A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & & \cdots\\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}\end{bmatrix}
$$

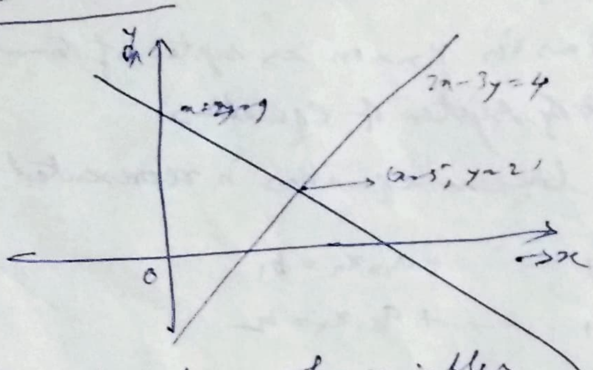
When we are going to solve the system of equations we are interested in identifying all values which satisfies all the equations in the system simultaneously.

There are four possibilities when we are going to solve a system of equations.

1. System has a unique solution
2. System has no solution
3. System has a solution but not unique.
4. System is in ill-condition.

~~5. Definite solution~~

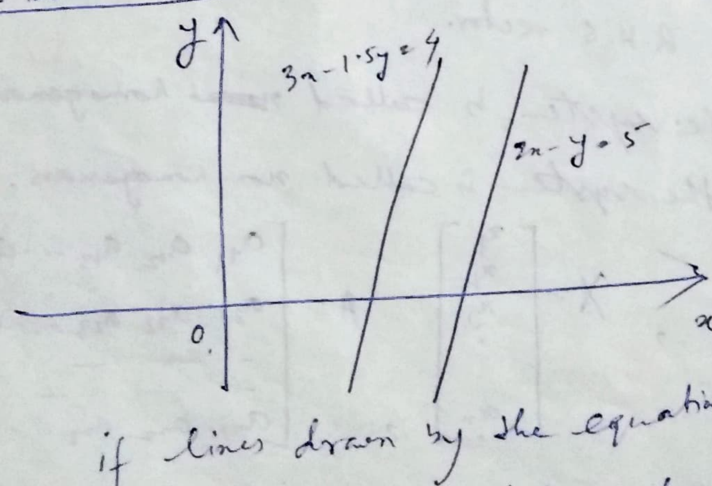
Unique solution



x=5, y=2.

When total no. of variables and, total no of equations are same in the system, and the lines drawn with these equations are intersecting in nature and system is not in ill-conditioned then the system have a unique solution.

No ~~soli~~ solution.

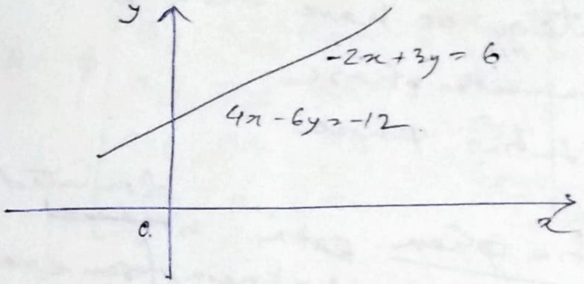


If lines drawn by the equations are parallel in nature, they cannot intersect to each other and never meet. These ~~equal~~ set of equations have no solutions also called inconsistent set of equations.

If no. of equations is greater than no. of variable then also this thing can be occurred.

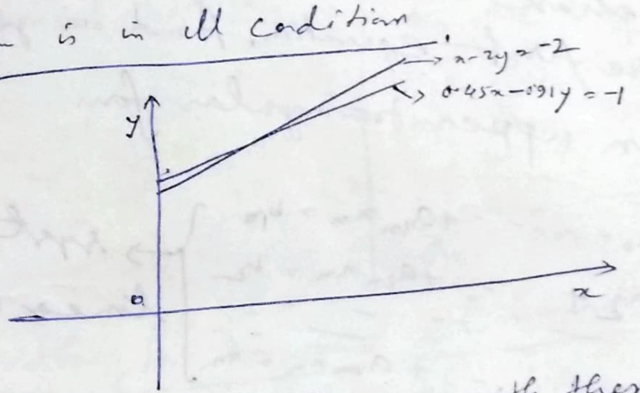
No unique or infinite solution

if the no of equations are less than the total no of variables then that system have more than one, or may be infinite solutions.



Rank (A) ≠ Rank A' = inconsistent.
no of variables

System is in ill condition



if line drawn with these equations are very closed to each other and we cannot identify the exact point of intersection easily then that system is called ill conditioned system.

The methods available for solving linear system of equation are basically of two types.

(i) Elimination approach
(ii) iterative approach

In elimination approach we have methods

1. Gauss elimination method.
2. Gauss Jordan method.
3. LU factorization method.
4. Matrix Inversion method. etc.

In iterative approach we have methods

(i) Gauss-Jacobi method.
(ii) Gauss-Seidal method etc.

Basic Gauss elimination method.

In this method we reduce the system of equation in upper traingular form and then we try to solve the system of equation.

In this strategy we have two main parts.

1. Forward elimination phase
2. back substitution phase

Forward elimination phase

In this phase ~~extra~~ one unknown is eliminated from every equation given except the first equation. And we give the system an upper traingular form.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \to \text{system of linear equation}$$

is reduced to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \quad a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n = b'_2 \\ \quad\quad + a''_{33}x_3 + \cdots + a''_{3n}x_n = b''_3 \\ \quad\quad\quad \cdots \\ \quad\quad\quad a_{n-1,n-1}x_{n-1} + a^{(n-1)}_{nn}x_n = b^{(n-1)}_n \end{array}\right\} \text{upper traingular form,}$$

This process is called forward elimination process.

Back substitution.

In this method we start from last equation and obtain the value of $x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$ and put it in previous eqn, and get the value of x_{n-1} next we put the value of x_n, x_{n-1} in its previous equation and get the value of x_{n-2} and so on.

This method is called back substitution.

Algorithm

System of linear equation can be represented by

$AX = B$

where A is coefficient matrix

i.e.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

augmented matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & & & & & b_n \\ & & & & a_{nn} & \\ a_{n1} & a_{n2} & a_{n3} & & & \end{bmatrix}$$

Algorithm for forward elimination process.

1. Take input the coefficient matrix of order '$n \times n$' as $a[i][j]$.
2. Take input the RHS vector of order '$(n \times 1)$' as $b[i]$.
3. make augmented matrix of $a[i][j]$ and $b[i]$ as $A[i][j]$.
4. for ($i = 1$ to $n-1$) do, i is incremented by 1.
5. ~~set temp~~ for ($j = i+1$ to n) do, j is incremented by 1.
6. temp = a_{ji}/a_{ii}
7. for ($k = i$ to $n+1$) do, k is incremented by 1.
8. set $a_{jk} = a_{jk} - \text{temp} \cdot a_{ik}$
9. endfor.
10. endfor
11. endfor,

$\frac{n(n+1)}{2}(n-1)$

$\frac{n(n+1)}{2}[(n-1)^n - \frac{n(n-1)}{2}] + \frac{n(n+1)}{2}(n-1) + \frac{n(n+1)}{2}(n-2) \cdots$

$\frac{n(n+1)}{2} \cdot \frac{n(n-1)}{2}$

$= \frac{n^2}{4}(n^2-1)$

Algorithm for backward substitution

1. set $x_n = b_n / a_{nn}$
2. for $i = (n-1)$ to 1 do, i is decremented by 1.
3. $x_i = b_i$
4. for $j = i+1$ to n do incremented by 1.
5. set $x_i = x_i - a_{ij} \cdot x_j$
6. endfor.
7. set $x_i = x_i / a_{ii}$
8. endfor
9. for $(i=1$ to $n)$ print x_i. i is incremented by 1.
10. endfor.
11. exit.

Problem

Solve the following system

$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16.$$

Ans The system can be represented as

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

i.e. $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$

~~The~~ Augmented matrix is,

$$A|b = \begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix}$$

now, row no. '1' is multiplied by $3/2$ and this result is subtracted from row no. 2 we have

$$\begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 1 & 4 & 9 & 16 \end{bmatrix}$$

now row no. 1 is multiplied by $1/2$ and result subtracted from row no. '3'.

$$\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 7/2 & 17/2 & 11 \end{bmatrix}$$

now, row no 2, is multiplied by 7 and result is subtracted from row no. 3

$$\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{bmatrix} \longrightarrow$$ matrix obtained after forward elimination.

$\therefore$ $$\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} -$$ Augmented matrix

$\therefore$ $$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ -10 \end{bmatrix}$$

$\therefore$ i.e.
$$\left.\begin{aligned} 2x + y + z &= 10 \\ \tfrac{1}{2}y + \tfrac{3}{2}z &= 3 \\ -2z &= -10 \end{aligned}\right\}$$

by solving this we have $z = \frac{-10}{-2} = 5$

putting $z=5$ in eqn '2' we have

$$\frac{1}{2}y + \frac{3}{2}\cdot 5 = 3$$

$$\therefore \frac{1}{2}y = 3 - \frac{3}{2}\cdot 5$$

$$\therefore \frac{1}{2}y = 3 - \frac{15}{2} = \frac{6-15}{2}$$

$$\therefore \cancel{\frac{1}{2}}y = \cancel{-}\frac{9}{\cancel{2}} \quad \therefore y=-9,$$

now putting $y=-9$, $z=5$ in eqn (1)

$$2x - 9 + 5 = 10$$

$$\text{or } 2x = 10 + 9 - 5 = 14$$

$$\therefore x = \cancel{\frac{14}{2}}^{7} = 7.$$

$$\therefore x = 7.$$

So the solution of the system is

$$\left.\begin{array}{l} x = 7 \\ y = -9 \\ z = 5 \end{array}\right\}. \text{ getting by back substitution}$$

The key lines are

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{k-1}}{a_{kk}^{k-1}} \cdot a_{kj}^{(k-1)}.$$

and

$$x_k = \frac{1}{a_{kk}^{(k-1)}}\left[b_k^{(k-1)} - \sum_{j=k+1}^{n} a_{kj}^{k-1} x_j \right],$$

Problem, Solve the following system

$$3x_1 + 6x_2 + x_3 = 16$$

$$2x_1 + 4x_2 + 3x_3 = 13$$

$$x_1 + 3x_2 + 2x_3 = 9$$

Ans $\left.\begin{array}{l} x_3 = 1 \\ x_2 = 2 \\ x_1 = 1 \end{array}\right\}$

Computational effort / Operation Count

In forward ~~the~~ elimination process,

for inner most loop.

multiplication = 1, subtraction = 1

loop circulates $(n+1)$ times

So. for inner most loop

multiplication = $n+1$, Subtraction = $n+1$ } for each j.

middle ~~loop~~ is $j = i+1$ to n.

outer loop is $i = 1$ to $(n-1)$

So, j varies from 2 to n

then 3 to n

—

n to n

So middle loop circulates at first $(n-1)$ times

then $(n-2)$ times

— — —

— 1 times.

So for middle loop at outer loop, $(i=1, j=2 \text{ to } n)$

$$\text{multiplication} = (n+1)(n-1) + (n+1)(n-2) + \cdots (n+1)\cdot 1$$
$$= (n+1)\{(n-1) + (n-2) + \cdots + 1\}$$
$$= (n+1)\cdot\frac{n(n-1)}{2} = \frac{1}{2}n(n^2-1).$$

In forward elimination method

for ~~$i=1$~~, $j = 2$ to n, $K = 1$ to $n+1$

for $i=2$, $j = 3$ to n, $K = 2$ ~~to~~

total no of multiplication = $(n-i+1)$ for inner most loop

and inner most loop circulates $(n-i)$ times.

where $i = 1$ to $n-1$,

$$\text{So. multiplication} = \sum_{i=1}^{n-1}(n-i+1)(n-i)$$

$$\text{Subtraction} = \sum_{i=1}^{n-1}(n-i+1)(n-i)$$

$$\text{division} = \sum_{i=1}^{n-1}(n-i).$$

For division $= (n-1)+(n-2)+\cdots+3+2+1$

$= \frac{1}{2}n(n-1)$

$\frac{n(n+1)(2n+1)}{6}$

$n(n-1)(2n-1)$

Subtraction/multiplication

$$= \sum_{i=1}^{n-1}(n-i+1)(n-i)$$

$$= \sum_{i=1}^{n-1}\{(n-i)^2+(n-i)\}$$

$$= \sum_{i=1}^{n-1}\{n^2-2ni+i^2+n-i\}$$

$$= \sum_{i=1}^{n-1}\{n^2+n-(2n+1)i+i^2\}$$

$$= (n^2+n)\cdot\sum_{i=1}^{n-1}1-(2n+1)\cdot\sum_{i=1}^{n-1}i+\sum_{i=1}^{n-1}i^2$$

$$= n(n+1)\cdot(n-1)-(2n+1)\cdot\frac{n(n-1)}{2}+\frac{n(n-1)(2n-1)}{6}$$

$$= \frac{n(n-1)}{2}\left\{2(n+1)-(2n+1)+\frac{(2n-1)}{3}\right\}$$

$$= \frac{n(n-1)}{2}\left\{\cancel{2n}+2-\cancel{2n}-1+\frac{2n-1}{3}\right\}$$

$$= \frac{n(n-1)}{2}\cdot\left\{1+\frac{2n-1}{3}\right\}$$

$$= \frac{n(n-1)}{2}\cdot\frac{2n+2}{3}$$

$$= \frac{n(n-1)}{\cancel{2}}\cdot\frac{\cancel{2}(n+1)}{3}$$

$$= \frac{1}{3}n(n^2-1)$$

In back substitution process

total no of multiplication/subtraction

$$= \sum_{j=1}^{n-1}(n-j) = n\sum_{i=1}^{n-1} 1 - \sum_{i=1}^{n-1} i$$

$$= \cancel{(n-1)\sum_{i=1}^{n-1} 1} - \sum_{i=1}^{n-1} i = n(n-1) - \frac{n(n-1)}{2}$$

$$= \frac{1}{2}n(n-1).$$

$$\cancel{= (n-1)(n-1) - \frac{1}{2}n(n-1)}$$

$$\cancel{= \frac{(n-1)}{2}\{2n-2-n\}}$$

$$\cancel{= \frac{(n-1)(n-2)}{2}}$$

$$\cancel{= \frac{n^2}{1}}$$

for multiplication $= \frac{1}{2}n(n-1)$.

subtraction $= \frac{1}{2}n(n-1)$

division $= (n-1)$.

So the total multiplication / subtraction

$$= \frac{1}{2}n(n-1) + \frac{1}{3}n(n^2-1)$$

$$= \frac{1}{2}n(n-1)\left\{1 + \frac{2}{3}(n+1)\right\}$$

$$= \frac{1}{2}n(n-1)\cdot\frac{(3+2n+2)}{3}$$

$$= \frac{n(n-1)(2n+5)}{6}.$$

division $= \frac{1}{2}n(n-1) + (n-1) = (n-1)\left(\frac{n+2}{2}\right) = \frac{1}{2}(n-1)(n+2).$ Ans.

Matrix Inversion Method

The system of equation n represented as

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

$\cdots$

$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$

— This can be represented in matrix form is give below.

$Ax = B$

so by premultiplying both side by A^{-1} we have

$A^{-1}Ax = A^{-1}B$

$\therefore Ix = A^{-1}B$

$\therefore X = A^{-1}B$

where A^{-1} can be obtained by many methods.

and, $|A| \neq 0$, then we can compute A^{-1}.

formula $A^{-1} = \dfrac{\text{Adj } A}{|A|}$,

This is the conventional method for obtaining inverse of a matrix

~~Gauss Jordan an~~

Another method, i.e Numerical method is Gauss-Jordan's Matrix Inversion Method.

Gauss-Jordan Matrix Inversion Method.

Here the coefficient matrix 'A' is taken side by side with a unit matrix of same order. And we do either all row operations or all column operations on coefficient matrix so that it ill become a unit matrix, but same operations will also done on that unit matrix, then the second matrix will ultimately become the inverse of matrix A, Find the inverse of the matrix

$$A = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc:ccc} 5 & -2 & 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 & 1 & 0 \\ 4 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

~~Divide~~ ~~$R_3 \leftarrow R_3/4$~~ $R_1 \leftarrow R_1/5$

$$\cancel{\left[\begin{array}{ccc:ccc} 5 & -2 & 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1/4 & 0 & 0 & 0 & 1/4 \end{array}\right]}$$

~~$R_1 \leftarrow R_1 - 5R_3$~~

$$\cancel{\begin{bmatrix} 5 & -2 & 4 \\ -2 & & \end{bmatrix}}$$

$$\left[\begin{array}{ccc:ccc} 1 & -2/5 & 4/5 & 1/5 & 0 & 0 \\ -2 & 1 & 1 & 0 & 1 & 0 \\ 4 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 + 2R_1$

$R_3 \leftarrow R_3 - 4R_1$

$$\left[\begin{array}{ccc:ccc} 1 & -2/5 & 4/5 & 1/5 & 0 & 0 \\ 0 & 1/5 & 13/5 & 2/5 & 1 & 0 \\ 0 & 13/5 & -16/5 & -4/5 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow 5R_2$

$$\left[\begin{array}{ccc:ccc} 1 & -2/5 & 4/5 & 1/5 & 0 & 0 \\ 0 & 1 & 13 & 2 & 5 & 0 \\ 0 & 13/5 & -16/5 & -4/5 & 0 & 1 \end{array}\right]$$

$R_1 \leftarrow R_1 + \frac{2}{5}R_2$

$R_3 \leftarrow R_3 - \frac{13}{5}R_2$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 6 & 1 & 2 & 0 \\ 0 & 1 & 13 & 2 & 5 & 0 \\ 0 & 0 & -37 & 6 & -13 & 1 \end{array}\right]$$

$R_3 \leftarrow -\frac{1}{37}R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 6 & 1 & 2 & 0 \\ 0 & 1 & 13 & 2 & 5 & 0 \\ 0 & 0 & 1 & -6/37 & 13/37 & -1/37 \end{array}\right] \qquad \begin{array}{l} R_2 \leftarrow R_2 - 13R_3 \\ R_1 \leftarrow R_1 - 6R_3 \end{array}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 73/37 & -4/37 & 6/37 \\ 0 & 1 & 0 & 152/37 & 16/37 & 13/37 \\ 0 & 0 & 1 & -6/37 & 13/37 & -1/37 \end{array}\right]$$

$$= \frac{1}{37}\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 73 & -4 & 6 \\ 0 & 1 & 0 & 152 & 16 & 13 \\ 0 & 0 & 1 & -6 & 13 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \frac{1}{37}\begin{bmatrix} 73 & -4 & 6 \\ 152 & 16 & 13 \\ -6 & 13 & -1 \end{bmatrix}.$$

Compute the values of the unknown in the system of equations by Gauss-Jordan's matrix Inversion Method.

1.
$$x_1 + 3x_2 + 2x_3 = 17$$
$$x_1 + 2x_2 + 3x_3 = 16$$
$$2x_1 - x_2 + 4x_3 = 13$$

Ans.
$$A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 17 \\ 16 \\ 13 \end{bmatrix}$$

$\therefore$ ~~Augmented matrix~~

$$A'B = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} 17 \\ 16 \\ 13 \end{bmatrix}$$

$\therefore$ Augmented matrix

$$A' = \left[\begin{array}{ccc:ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & -1 & 4 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - 2R_1 \end{array}$$

we have.

$$A' \cong \left[\begin{array}{ccc:ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -7 & 0 & -2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow 3R_2 + R_1 \\ R_3 \leftarrow R_3 - 7R_2 \end{array}$$

$$\cong \left[\begin{array}{ccc:ccc} 1 & 0 & 5 & -2 & 3 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -7 & 5 & -7 & 1 \end{array}\right] \quad R_2 \leftarrow -R_2$$

$$\cong \left[\begin{array}{ccc:ccc} 1 & 0 & 5 & -2 & 3 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -7 & 5 & -7 & 1 \end{array}\right] \quad R_3 \leftarrow -R_3/7$$

$$\cong \left[\begin{array}{ccc:ccc} 1 & 0 & 5 & -2 & 3 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 5/7 & -1 & 1/7 \end{array}\right] \quad R_3 \leftarrow -R_3$$

$$\cong \left[\begin{array}{ccc:ccc} 1 & 0 & 5 & -2 & 3 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -5/7 & 1 & -1/7 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 + R_3 \\ R_1 \leftarrow R_1 - 5R_3 \end{array}$$

$$\cong \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 11/7 & -2 & 5/7 \\ 0 & 1 & 0 & 2/7 & 0 & -1/7 \\ 0 & 0 & 1 & -5/7 & 1 & -1/7 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 11/7 & -2 & 5/7 \\ 2/7 & 0 & -1/7 \\ -5/7 & 1 & -1/7 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \begin{bmatrix} 11/7 & -2 & 5/7 \\ 2/7 & 0 & -1/7 \\ -5/7 & 1 & -1/7 \end{bmatrix} \begin{bmatrix} 17 \\ 16 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11}{7}\cdot 17 - 2\cdot 16 + \frac{5}{7}\cdot 13 \\ \frac{2}{7}\cdot 17 + 0\cdot 16 - \frac{1}{7}\cdot 13 \\ -\frac{5}{7}\cdot 17 + 1\cdot 16 - \frac{1}{7}\cdot 13 \end{bmatrix} = \begin{bmatrix} \frac{187}{7} + \frac{65}{7} - 32 \\ \frac{34}{7} - 13/7 \\ 16 - \frac{85}{7} - \frac{13}{7} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{187 + 65 - 224}{7} \\ \frac{34 - 13}{7} \\ \frac{112 - 85 - 13}{7} \end{bmatrix} = \begin{bmatrix} \frac{252 - 224}{7} \\ \frac{21}{7} \\ \frac{112 - 98}{7} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$\therefore X = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ $\therefore x_1 = 4,\ x_2 = 3,\ x_3 = 2$ Ans.

problem.
$$\left.\begin{aligned} 8x_1 + 2x_2 - 2x_3 &= 8 \\ x_1 - 8x_2 + 3x_3 &= -4 \\ 2x_1 + x_2 + 9x_3 &= 12 \end{aligned}\right\} \text{Ans } x_1 = x_2 = x_3 = 1,$$

Algorithm.

Algorithm for inverse

1. Take input the coefficient matrix $A[i,j]$, of order $n \times n$.
   Take R.H.S vector $B[i]$, of order $(n \times 1)$.
2. Take ~~another~~ ~~no~~ unit matrix $U[i,j]$ of order $n \times n$.
3. for ($i=1$ to n) do. i is incremented by 1.
4. for ($j=1$ to n) do j is incremented by 1
   where $j \neq i$
5. temp $= a_{ji}/a_{ii}$
6. for ($k = i$ to n) do, k is incremented by 1
7. set $a_{jk} = a_{jk} -$ temp $* a_{ik}$.
   7a. set $u_{jk} = u_{jk} -$ temp $* a_{ik}$.
8. end for.
9. end for,
10. end for.
11. for ($i=1$ to n) i is incremented by 1.
12. for ($j=1$ to n) j is incremented by 1
13. set $u_{ij} = u_{ij} / \cancel{A[i][i]}\ a_{ii}$
14. set $a_{ii} = a_{ii} / \cancel{A[i][i]}\ a_{ii}$
15. end for
16. end for,
17. exit.

Computational effort

multiplication and subtraction $\Big\} = \sum_{i=1}^{n} 2(n-(i+1)).n$

$$= \sum_{i=1}^{n} (2n^2+2n-2ni)$$

$$= (2n^2+2n) \cdot \sum_{i=1}^{n} 1 - 2n \sum_{i=1}^{n} i$$

$$= (2n^2+2n) \cdot n - 2n \cdot \frac{n(n+1)}{2}$$

$$= 2n^3+2n^2-n^3-n^2$$

$$= n^3+n^2 = n^2(n+1).$$

division $= n^2+2n^2 = 3n^2.$

$\therefore$ division $= 3n^2$

Algorithm for obtaining solutions.

(18). for i=1 to n do. i is incremented by 1
(18a) set, sum=0.
(19) for j=1 to n do j is incremented by 1
(20) set sum = sum + a[i][j] * B[j]
(21) endfor.
(22) set X[i] = sum;
(23) endfor.
(24) for(i=1 to n) print x[i] as results.
(25) ~~exit~~ endfor.
(26) exit.

no. of addition and multiplication needed $= n^2$

So total multiplication needed $= n^2(n+1)+n^2$
$= n^2(n+2)$.

" addition " $= n^2$

" substraction " $= n^2(n+1)$

" division " $= 3n^2$,

L.U. factorization Method

if the coefficient matrix be $A[i][j]$ for system of eqⁿs 'L' and 'U' if two factors of A then.

$$A = L.U$$

Here L is ~~upper~~ lower traingular matrix

and U " upper traingular matrix

So.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & & l_{nn} \end{bmatrix}$$

and

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

From system of equation we know that

$$AX = B.$$

then $(LU)X = B$

$$L(UX) = B$$

let $UX = Z$

therefore $LZ = B$.

So we can obtain Z, by forward substitution process

and " " " X " backward " " .

The elements of L and U are determined, by comparing the elements of the product of L and U with those of A.

therefore

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & 0 & \cdots & 0 \\ \cdots & & & & & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & & l_{nn} \end{bmatrix} \times \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \cdots & & & & \\ 0 & 0 & 0 & & u_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

or

$$\begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} & \cdots & l_{11}u_{1n} \\ l_{21}u_{11} & l_{21}u_{12}+l_{22}u_{22} & l_{21}u_{13}+l_{22}u_{23} & \cdots & l_{21}u_{1n}+l_{22}u_{2n} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+l_{33}u_{33} & \cdots & l_{31}u_{1n}+l_{32}u_{2n}+l_{33}u_{3n} \\ \vdots & & & & \\ l_{n1}u_{11} & l_{n1}u_{12}+l_{n2}u_{22} & l_{n1}u_{13}+l_{n2}u_{23}+l_{n3}u_{33} & \cdots & l_{n1}u_{1n}+l_{n2}u_{2n}+\cdots+l_{nn}u_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

So by equalizing these two matrices we have total of $(n \times n) = n^2$ equations.

and total of $2 \times \frac{n(n+1)}{2}$ [each of L and U contains $(1+2+3+\cdots+n)$ no. of unknowns]

$= n(n+1) = n^2+n$ unknowns

So as the total no. of equations is less than the total no of ~~eq~~ unknowns, so the solutions are not unique.

So to produce unique solutions we have to reduce the number of unknowns to n^2.

It is done by assuming diagonal elements of L or U to be 'One'.

The decomposition with L having unit diagonal value is called Dolittle LU Decomposition method.

And otherwise
The decomposition with U having unit diagonal value is called the Crouts LU Decomposition method.

The crouts Algorithm:

In this method, diagonal values of U matrix is considered as '1'.

$$\begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \times \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & u_{24} & \cdots & u_{2n} \\ 0 & 0 & 1 & u_{34} & \cdots & u_{3n} \\ \vdots & & & 1 & & \vdots \\ \vdots & & & & & 1 \\ 0 & 0 & & & & \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

So

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} & l_{11}u_{14} & \cdots \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} & l_{21}u_{14}+l_{22}u_{24} & \cdots \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} & l_{31}u_{14}+l_{32}u_{24}+l_{33}u_{34} & \cdots \\ l_{41} & l_{41}u_{12}+l_{42} & l_{41}u_{13}+l_{42}u_{23}+l_{43} & l_{41}u_{14}+l_{42}u_{24}+l_{43}u_{34}+l_{44} & \cdots \\ \vdots & \vdots & \vdots & & \\ l_{n1} & l_{n1}u_{12}+l_{n2} & l_{n1}u_{13}+l_{n2}u_{23}+l_{n3} & l_{n1}u_{14}+l_{n2}u_{24}+l_{n3}u_{34}+l_{n4} & \cdots \end{bmatrix}$$

$$\begin{bmatrix} \cdots & l_{11}u_{1n} \\ \cdots & l_{21}u_{1n}+l_{22}u_{2n} \\ \cdots & l_{31}u_{1n}+l_{32}u_{2n}+l_{33}u_{3n} \\ \cdots & \\ \cdots & l_{n1}u_{1n}+l_{n2}u_{2n}+l_{n3}u_{3n}+\cdots+l_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots & a_{3n} \\ a_{41} & a_{42} & a_{43} \; a_{44} \cdots & a_{4n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

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Then the algorithm's approach is similar to Dolittle algorithm.
in this crouts algorithm, L and U matrix are of the form

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & & & & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Here $u_{ii} = 1$ [where $i = 1$ to n],

and total no. of unknown here $= \frac{1}{2}n(n+1) + \frac{1}{2}n(n-1)$

$= \frac{1}{2}n(n+1+n-1) = \frac{1}{2} n \cdot 2n$

$= n^2$

here n^2 eqn and n^2 variables,
we can obtain value of knowns by the following formula.

if $(i \geq j)$ and $(j \neq 1)$

$$a_{ij} = \cancel{l_{i1} \cdot u_{1j} + l_{ij}} \left[\sum_{k=1}^{j-1} l_{ik} \cdot u_{kj}\right] + [l_{ij}]$$

here $j = 2$ to i.
and $i = 2$ to n.

if $(j = 1)$

$l_{i1} = a_{i1} \rightarrow i = 1$ to n. $\quad l_{11} u_{1j} = a_{1j}$ [$j = 2$ to n]

if $(i < j)$ and $(i = 1)$

if $(i < j)$ and $(i \neq 1)$. here $j = 2$ to n
and $i = 2$ to $j-1$,

$$a_{ij} = \sum_{k=1}^{i} l_{ik} u_{kj}$$

where $i = 1$ to n $\quad u_{ii} = 1$.

$u_{ij} = 0$. ~~and~~

if $(i > j)$

$l_{ij} = 0$

if $(i < j)$

$$a_{23} = \sum_{k=1}^{i} l_{21} u_{13} + l_{22} u_{23}$$

Algorithm for crouts method and solution

1. Given n, A, b
2. for $i=1$ to n do, $i=i+1$
3. set $l_{i1} = a_{i1}$, (3a) endfor
4. for $i=1$ to n do, $i=i+1$
5. set $u_{ii} = 1$, (5a) endfor
6. for $j=2$ to n do, $j=j+1$
7. set $u_{1j} = a_{1j}/l_{11}$, ~~—~~ (7a) endfor.
8. for each $i=2$ to n do $i=i+1$
   (a) for $j=2$ to i do, $j=j+1$
   (b) set $l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} \cdot u_{kj}$
   (c) endfor.
   (d) for $j=2$ to $i-1$ do, $j=j+1$
   (e) set $u_{ji} = \cancel{\sum_{k=1}^{j} l_{jk}\cdot u_{ki}}\ \frac{1}{l_{jj}}\left[a_{ji} - \sum_{k=1}^{j-1} l_{jk}\cdot u_{ki}\right]$
   (f) endfor.
9. endfor.

Solution

10. Set $z_1 = b_1$
11. for $i=2$ to n do $i=i+1$
12. set $sum = \sum_{j=1}^{i-1} l_{ij} z_j$
13. set $z_i = b_i - sum$.
14. ~~endfor~~ set $z_i = z_i / l_{ii}$
15. ~~set~~ endfor
16. set $x_n = z_n$
17. for $i=n-1$ to 1 do $i=i-1$
18. Set $sum = \sum_{j=i+1}^{n} u_{ij} x_j$
19. set $x_i = b_i - sum$.
20. endfor.
21. Write results
22. exit

problem

Solve the system

$3x_1 + 2x_2 + x_3 = 10$

$2x_1 + 3x_2 + 2x_3 = 14$

$x_1 + 2x_2 + 3x_3 = 14$

Use Crout's LU decomposition method.

Ans Here coefficient matrix

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

now.

$A = LU$

$$\therefore \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \times \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

now, $l_{11} = 3$, $l_{21} = 2$, $l_{31} = 1$

now $\left.\begin{aligned} l_{11} \cdot u_{12} &= 2 \\ l_{11} \cdot u_{13} &= 1 \end{aligned}\right\}$ $\therefore u_{12} = 2/3$ and $u_{13} = 1/3$

next $l_{21} u_{12} + l_{22} = a_{22}$

$\therefore 2 \cdot u_{12} + l_{22} = 3$

$\Rightarrow 2 \cdot 2/3 + l_{22} = 3$

$\therefore l_{22} = 3 - 4/3 = 5/3$

next $l_{31} \cdot u_{12} + l_{32} = a_{32}$

$\Rightarrow 1 \cdot 2/3 + l_{32} = 2$

$\therefore l_{32} = 2 - 2/3 = 4/3$

next $l_{21} \cdot u_{13} + l_{22} \cdot u_{23} = a_{23}$

or. $2 \cdot \frac{1}{3} + \frac{5}{3} \cdot u_{23} = 2$

or $\frac{2}{3} + 5/3 \cdot u_{23} = 2$

or $\frac{5}{3} \cdot u_{23} = 2 - 2/3 = 4/3$

$\therefore \quad u_{23} = 4/5$

last: $l_{31} \cdot u_{13} + l_{32} \cdot u_{23} + l_{33} = a_{33}$

$1 \cdot \frac{1}{3} + \frac{4}{3} \cdot \frac{4}{5} + l_{33} = 3$

or $\frac{1}{3} + \frac{16}{15} + l_{33} = 3$

or $\frac{5+16}{15} + l_{33} = 3$

or $l_{33} = 3 - \cancel{\frac{21}{15}}\frac{7}{5} = 3 - 7/5 = 8/5$

$\therefore \quad l_{33} = 8/5$

So, the matrices will become

$$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 5/3 & 0 \\ 1 & 4/3 & 8/5 \end{bmatrix} \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Now. we know that

$LZ = B.$

$$\therefore \begin{bmatrix} 3 & 0 & 0 \\ 2 & 5/3 & 0 \\ 1 & 4/3 & 8/5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$\therefore \; 3z_1 = 10$

$\therefore \; z_1 = 10/3 ,$

$2z_1 + 5/3 \, z_2 = 14.$

or. $2 \cdot 10/3 + 5/3 \cdot z_2 = 14$

or. $5/3 \cdot z_2 = 14 - \frac{20}{3}$

or. $5/3 \cdot z_2 = \frac{42-20}{3} = 22/3$

or. $5/\cancel{3} \cdot z_2 = 22/\cancel{3}$

$\therefore \; z_2 = 22/5$

next $\quad z_1 + \frac{4}{3} \cdot z_2 + 8/5 \cdot z_3 = 14.$

or. $\frac{10}{3} + \frac{4}{3} \cdot \frac{22}{5} + 8/5 \cdot z_3 = 14$

or. $\frac{10}{3} + \frac{88}{15} + 8/5 \cdot z_3 = 14$

or. $\frac{50+88}{15} + 8/5 \cdot z_3 = 14$

or. $\frac{138}{15} + 8/5 \cdot z_3 = 14$

or $\quad 8/5 \cdot z_3 = 14 - \frac{138}{15} = \frac{210-138}{15}$

or $\quad \cancel{8}/\cancel{5} \cdot z_3 = \cancel{72}^{24\,3}/\cancel{15}_{\cancel{5}}$ $\quad \therefore \; z_3 = 3.$

$$\therefore \; z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 22/5 \\ 3 \end{bmatrix}$$

now we know that, $\quad Ux = z$

$$\therefore \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 22/5 \\ 3 \end{bmatrix}$$

$\therefore \; x_3 = 3 ,$ $\quad x_2 + 4/5 \cdot x_3 = 22/5$