Numerical solution of Algebraic and trancendental equation

In engineering and scientific works, in most of the cases we have to find the value of x, in the equation given of the form.

$$f(x) = 0,$$

where the values of x's are called roots or zero's of that equation. if this equation is linear we can easily get the value of x,

if it is non linear,

Then if the equation is a polynomial equation of degree two or three or four, then we have the different formula's for solving this equation.

But if the polynomial equation is degree greater than that or if it is a trancendental equation then there is no Direct formula for solving that type of equations.

Eg:- $y = 5x + 3 = f(x)$

$f(x) = 0$ is linear equation.

if $f(x) = 5x^2 + 3 = 0$. then it is non linear polynomial equation.

may be $f(x) = 5x^4 + 3x^3 + 2x^2 + 5 = 0$. → non linear polynomial eqn

if $f(x) = x^n + e^x = 0$ → This type of eqn's can be called as transcendental equation.

— So for this type of equations we have no direct method and we need an approximate method to solve it.

There are no. of ways to find the roots of nonlinear equations - They are.

1. Direct method.
2. Graphical method
3. Traial and error method,
4. Iterative method.

Among them here we study only iterative methods. These methods are basically divided into two groups.

① Bracketing methods ⇒ which starts with two initial guess values which brackets the root.

(a) Bisection method.

(b) Regula-Falsi method.

②. Open end method, which starts with only single guess value.

(a) Newton-Raphson Method

(b) Secant method

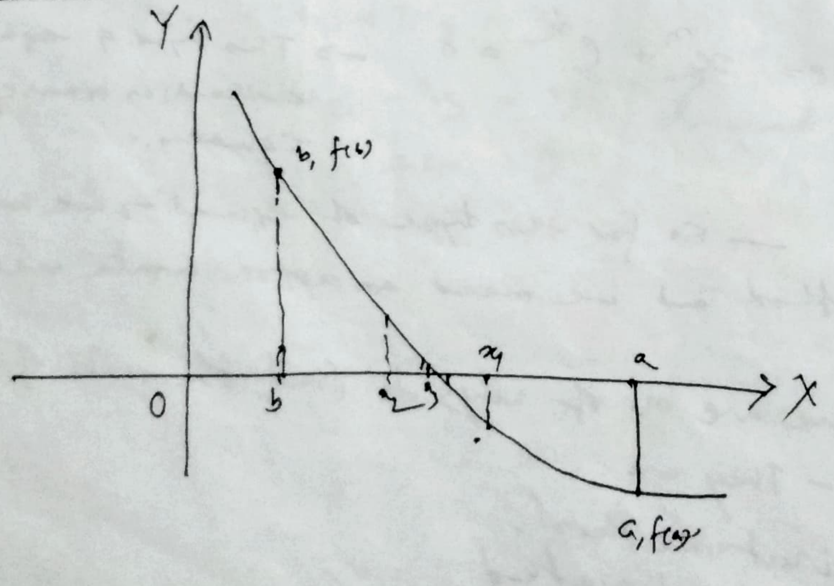
(c) Muller's method.

In case of bracketing methods we take two initial guess values, say ~~a and~~ a and b, this value are such that $f(a) \times f(b) < 0$, condition should occur otherwise the root will not lie within the interval a to b.

This rule is called descartey's rule of sign.

And we stop the iteration if we achieve a relative error less than that of a certain percentage between two successive iterative values of x.

Bisection method.



In this method we start with ~~the~~ two initial guess

$x_1 = a$, $x_2 = b$

such that $f(a) \times f(b) < 0$

then root x lies between x_1 and x_2

rather a and b.

new, x is computed as $x = \frac{x_1 + x_2}{2}$

then. if $f(x) \times f(x_1) < 0$ then the $[x, x_1]$ interval

contains the root, otherwise $[x, x_2]$ interval contains

the root.

Algorithm

1. Take input the function $f(x)$
2. Take input the limits of the root x_1 and x_2
3. check, if $f(x_1) \times f(x_2) < 0$,
if not goto step 2 again and 'take new set of values'.

~~4. if $f(x_1) \times f(x_2) < 0$ then.~~

4. Compute $x = (x_1 + x_2)/2$, (4a) set $temp = x$.

5. if $f(x) \times f(x_1) < 0$ then $[x, x_1$ contains root$]$

6. set $x_2 = x$

otherwise

7. set $x_1 = x$

8. set $x = \frac{x_1 + x_2}{2}$

9. if $\left(\left|\frac{temp - x}{temp}\right|\right)$ is greater than error,

goto step (4a).

otherwise

write the value of 'x' and exit.

10. stop.

problem Find the root of the equation

$x^2 - 4x - 10 = 0$, use bisection method.

Ans let $x_1 = 5$, $x_2 = 6$.

because $x_1^2 - 4x_1 - 10 = 25 - 20 - 10 = -5$

$\therefore f(x_1) = -5$

and $x_2^2 - 4x_2 - 10 = 36 - 24 - 10 = 2$,

$\therefore f(x_2) = 2$.

$\therefore f(x_1) \times f(x_2) = 2 \times (-5) = -10$.

$\therefore$ root lies between x_1 and x_2

now, $x = \frac{x_1 + x_2}{2} = \frac{5+6}{2} = \frac{11}{2} = 5.5$

now $f(x) = (5.5)^2 - 4(5.5) - 10$

$= 30.25 - 22 - 10$

$= 30.25 - 32$

$= -1.75$

now, previous $f(x_1) = -5$, and $f(x_2) = 2$.

so, $f(x_1) \times f(x) > 0$,

$\therefore$ root lies between x and x_2

so here $x_1 = 5.5$, $x_2 = 6$.

$\therefore$ new $x = \frac{x_1 + x_2}{2} = \frac{5.50 + 6.00}{2} = 5.75$

now $f(x) = (5.75)^2 - 4.5.75 - 10$

$= 33.0625 - 23 - 10$

$= 33.0625 - 33$

$= .0625$

now, root lies between ~~x_1~~ 5.5 and $x = 5.75$

$\therefore$ now $x_1 = 5.5$, $x_2 = 5.75$

$\therefore$ new $x = \frac{5.50 + 5.75}{2} = 5.625$

So upto two decimal places the root between interval "5 and 6" is given $= 5.62$ Ans.

Problem Find a real root of the equation

$f(x) = x^3 - x - 1 = 0$.

Convergence of bisection method

In bisection method, we start with two initial guess x_1 and x_2, and in every iteration the step is reduced to the size by half.

Then after n repetition the interval are will become

$$\frac{x_2 - x_1}{2^n} = \frac{\Delta x}{2^n}$$

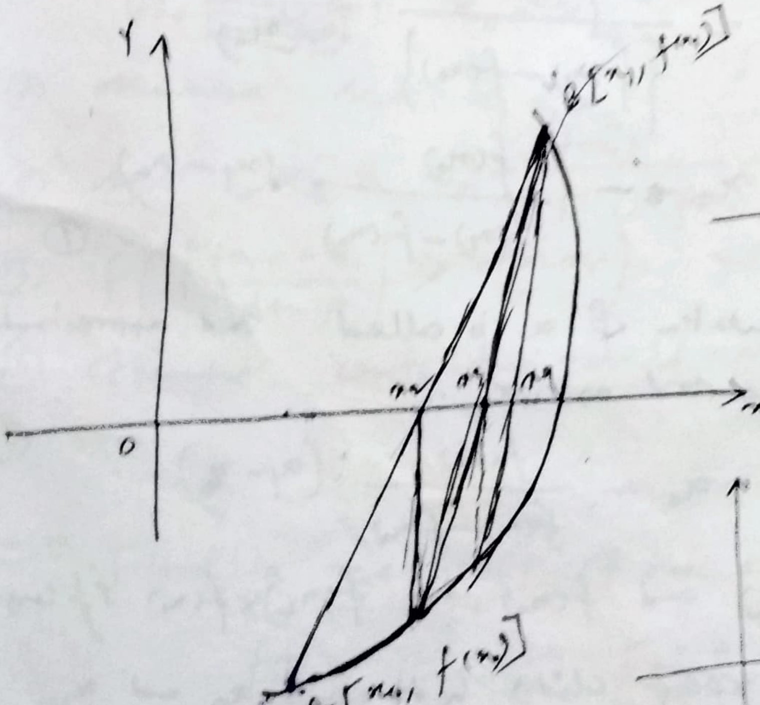
After n iteration the root must lie within $\pm \frac{\Delta x}{2^{n+1}}$

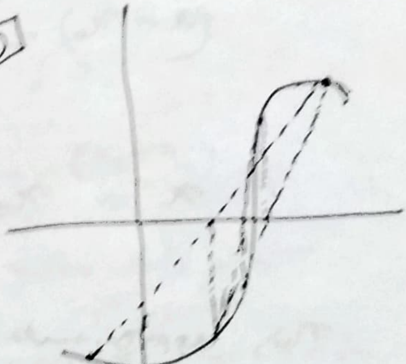
So. $E_n = \left| \frac{\Delta x}{2^{n+1}} \right|$

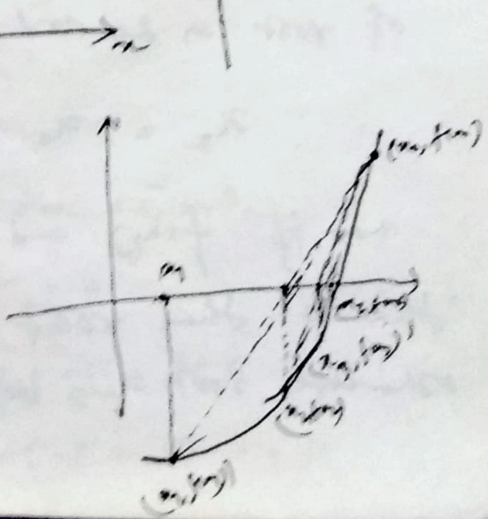
Similarly $E_{n+1} = \left| \frac{\Delta x}{2^{n+2}} \right| = E_n/2$.

So error is divided in every step by a factor of 2. Hence this method is linearly convergent. Here due to slow convergence here we need large number of iterations to achieve high degree of accuracy.

Method of false position.







In the bisection method we ~~starts~~ with two equal halves, but in this method we achieve the points more closer to root than previous method.

Say two initial ~~gu~~ guesses are $\{x_1, f(x_1)\}$ and $\{x_0, f(x_0)\}$

For getting the solution we at $y=0$, we ~~act~~ actually replace the curve $f(x)$ by the chord, joing this two points. And from the equation of the chord we can have the solution. Here in this method we use the false position of the root repeatedly, that is why the method is called method of false position.

Then the equation of the chord is,

$$\frac{y - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$$

$$\therefore (x - x_0) = \frac{(y - f(x_0))(x_1 - x_0)}{\cancel{(x - x_0)}} \times \frac{1}{\{f(x_1) - f(x_0)\}}$$

putting in this equation $y=0$ we have

$$(x - x_0) = \frac{0 - f(x_0)}{\{f(x_1) - f(x_0)\}} \cdot \frac{(x_1 - x_0)}{\cancel{(x - x_0)}}$$

$$\therefore x = x_0 - \frac{f(x_0)}{f(x_1) - f(x_0)} \cdot (x_1 - x_0) \quad \ldots \text{①}$$

This approximatation of x is called 2nd approximation of root as denoted as x_2

$$\therefore x_2 = x_0 - \frac{f(x_0)}{f(x_1) - f(x_0)} \cdot (x_1 - x_0)$$

now if $f(x_2)$ and $f(x_0)$, i.e $f(x_2) \times f(x_0)$ if less than '0' then root lies between x_0 and x_2 otherwise root lies between (x_2 and x_1)

we will get next approximation.

in first case, then x_2's value is assigned in 'x_1'
otherwise.
in 2nd case,

x_2's value is assigned in x_0.

then we have new set of $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$
with the help of this we will get the new approximation value 'x_2'

Algorithm.

1. Take input the function $f(x)$
2. Take two initial guess x_1 and x_0
3. check if $f(x_1) \times f(x_0) < 0$,
   if not go to step 2 and take new set of values.
4. Compute

$$x_2 = x_0 - \frac{f(x_0)}{f(x_1) - f(x_0)} \cdot (x_1 - x_0)$$

5. set temp $= x_2$
6. if $f(x_2) \times f(x_1) < 0$,
   then set $x_0 = x_2$.
7. otherwise set $x_1 = x_2$
8. set $x_2 = x_0 - \frac{f(x_0)}{f(x_1) - f(x_0)} \cdot (x_1 - x_0)$
9. if $\left(\left|\frac{temp - x}{temp}\right| > error\right)$ goto step 5
10. otherwise, write the result as 'x_2' and exit.
11. Stop.

Problem:- Find a real root of the equation

$$f(x) = x^3 - 2x - 5 = 0.$$

Ans. from the eqn. $f(2) = 2^3 - 2.2 - 5 = 8 - 4 - 5 = -1$

and $f(3) = 3^3 - 2.3 - 5 = 27 - 6 - 5 = 27 - 11 = 16.$

So we have $f(2) = -1$ and $f(3) = 16$.

So the root lies between $x_0 = 2$, and $x_1 = 3$.

now by using the formula for regula-falsi method we have.

$$x_2 = x_0 - \frac{f(x_0)}{f(x_1) - f(x_0)} \cdot (x_1 - x_0)$$

$$= 2 - \frac{f(2)}{f(3) - f(2)} \cdot (3-2).$$

$$= 2 - \frac{-1}{16-(-1)} \cdot 1 = 2 + \frac{1}{16+1} = 2 + \frac{1}{17}$$

$\therefore x_2 = \frac{35}{17} = 2.0588235$

now, $f(x_2) = (x_2)^3 - 2 \cdot x_2 - 5$

$= 8.7268468 - \cancel{2}\ 4.1176470 - 5$

$= 8.7268468 - 9.1176470$

$= -0.3908002.$

5435

5435

$\therefore f(x_2) < 0.$

$\therefore f(x_0) \times f(x_2) > 0$ [as $f(x_0) < 0$]

$\therefore$ root lies between x_2 and x_1.

now for algorithm. $x_0 = x_2 = 2.0588235$

$x_1 = 3.$

then new

$$x_2 = x_0 - \frac{f(x_0)}{f(x_1) - f(x_0)} \cdot (x_1 - x_0)$$

$$= 2.0588235 - \frac{-0.3908002}{16 - (-0.3908002)} \cdot (3 - 2.0588\ldots$$

$$= 2.0588235 + \frac{0.3908002}{16.3908002} \cdot 0.9411765$$

$= 2.0588235 + 0.0224401$

$= 2.0812636.$

by repeating that step after ~~five~~ four iteration we have the value

$$x_2 = 2.0934.$$

<u>problems</u>: solve the equation $x^3 - 9x + 1 = 0$, correct upto two decimal places, use method of false position.

<u>Convergency in false position method</u>

In this method, we start with ~~one~~ two initial guess values, which brackets the root. ~~Here one point is fixed and another point is moving~~ and subsequently compute the approximate root in every step.

so. at first step. $e_1 = x_1 - x_0$

then $e_2 = x_2 - x_1$

then $e_3 = x_3 - x_2$

$\vdots$

$e_{n+1} = (x_{n+1} - x_n).$

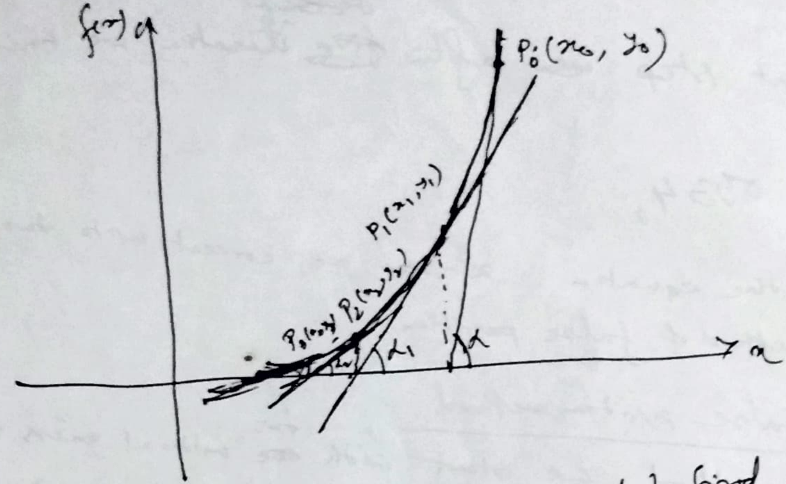
It can be shown that,

$$e_{n+1} = e_n \cdot \frac{(x_{n+1} - x_n) \cdot f''(R)}{f'(R)}.$$

So. if we fix one point, from beginning. then. x_n is fixed., $f''(R)$ and $f'(R)$ are constant for some R. then this is the linear relationship.

So, the iteration converges linearly in this case

Newton - Raphson Method:-



This is the method used to find isolated root of a equation $f(x) = 0$, Here we starts with a initial guess for root say x_0, and in successive iteration we try to ~~fo~~ get closure of exact root.

Say h is the small correction given by this method on initial guess x_0, take it as x_1

then $x_1 = x_0 + h$,

If x_0 is not the exact root,

then $f(x_0) \neq 0$,

say after a small correction $f(x_1) = 0$ (may be).

therefore $f(x_0 + h) = 0$

by using Taylor's series method we have

$$f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$$

here as h is very small, we can neglect 2nd or higher order term, then we have,

$$f(x_0) + h f'(x_0) + 0 + \cdots 0 = 0$$

$$\text{or, } f(x_0) + h f'(x_0) = 0$$

$$\therefore f(x_0) = -h f'(x_0) \quad \therefore h = -\frac{f(x_0)}{f'(x_0)}$$

therefore $x_1 = x_0 + h$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ [by putting the value of h]

similarly if we do some correction on 'x_1', say it is 'h_1'

then, if x_2 is the 2nd correction then we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

so in general we have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Here actually we start with an initial point (x_0, y_0) on the curve, then we draw a tangent at that point and it intersects x axis at some point x_1, then x_1 is the first approximation of $x_0, f(x_0)$ point. Then from $x_1, f(x_1)$ point we draw another tangent at that point and it intersects x axis at x_2, so x_2 is the 2nd approximation. thus we go closer to the root quickly.

Algorithm for Newton Raphson Method.

① Define $f(x)$ and $f'(x)$.

②. take an initial guess of the root as 'x_0'.

③. Evaluate $f(x_0)$ and $f'(x_0)$.

④ set $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ Er.

⑤ ~~compare~~ determine relative error in computing 'x_1' with respect to 'x_0'

~~set~~ set $E_r = \left|\frac{x_1 - x_0}{x_1}\right|$

⑥ if $E_r \leq$ error (a predefined error value)

then ~~exit~~ print the result as x_1 and exit.

Otherwise

⑦. ~~replace x_1 by the~~ set $x_0 = x_1$

and go to step 3.

⑧. exit.

Convergency in Newton Raphson method.

Let x_n be the nth approximation root of $f(x)$ function. Let x_{n+1} is very close to x_n. so.

$$x_{n+1} = x_n + h.$$

$$\therefore f(x_{n+1}) = f(x_n + h) = 0$$

$\therefore$ Expanding by taylor's series we have

$$f(x_n) + h.f'(x_n) + \frac{h^2}{2!} f''(x_n) + \cdots - \cancel{\ } = 0$$

by neglecting ~~2nd~~ 3rd or higher order terms we have.

$$= f(x_n) + h.f'(x_n) + \frac{h^2}{2!} f''(x_n) = 0.$$

~~Let exact root is x_r~~

~~$x_{n+1} = x_r$~~

and $h = x_{n+1} - x_n$

$$= x_r - x_n$$

So we have

$$f(x_n) + (x_r - x_n).f'(x_n) + \frac{1}{2} f''(x_n).(x_r - x_n)^2 - - \quad (1)$$

again we know that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\text{or, } x_r = x_n - \frac{f(x)}{f'(x_n)}$$

$$(x_r - x_{n+1}) + \frac{1}{2} \frac{f'(x_{n+1})}{f'(x_{n+1})}.($$

$$\text{or } (x_r - x_n) = - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore f'(x_n).(x_r + x_n) = - f(x_n)$$

$(x_r - x_n) \times y = x_{n+1} + x_r$

$x_{n+1} - x_n$

$- x_r + x_n$

$y = x_{n+1} - x_r - x_r + x_n$

$\therefore f(x_n) + (x_{n+1} - x_n) f'(x_n) + \frac{1}{2} f''(x_n) \cdot (x_{n+1} - x_n)^2 = 0$

Now If the exact root location say x_r

$\therefore e_n = (x_r - x_n)$

and $e_{n+1} = (x_r - x_{n+1})$

divide both side by $f'(x_n)$

we have

$$\frac{f(x_n)}{f'(x_n)} + (x_{n+1} - x_n) + \frac{1}{2} \frac{f''(x_n)}{f'(x_n)} \cdot (x_{n+1} - x_n)^2 = 0$$

$$\therefore (x_{n+1} - x_n) + \frac{1}{2} \frac{f''(x_n)}{f'(x_n)} \cdot (x_{n+1} - x_n)^2 = -\frac{f(x_n)}{f'(x_n)}$$

but we know that, then [Here we just replacing x_{n+1} with x_r]

if the exact root is x_r,

$$(x_r - x_n) + \frac{1}{2} \frac{f''(x_n)}{f'(x_n)} \cdot (x_r - x_n)^2 = -\frac{f(x_n)}{f'(x_n)}$$

let $e_n = |x_r - x_n|$

$$\therefore e_n + \frac{1}{2} \frac{f''(x_n)}{f'(x_n)} e_n^2 = -\frac{f(x_n)}{f'(x_n)}$$

$$\therefore \frac{1}{2} \cdot \frac{f''(x_n)}{f'(x_n)} \cdot e_n^2 \cong (x_{n+1} - x_n) - (x_r - x_n)$$

[as $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

and $e_n = (x_r - x_n)$]

$$\frac{1}{2} \frac{f''(x_n)}{f'(x_n)} \cdot e_n^2 = x_{n+1} \cancel{- x_n} - x_r \cancel{+ x_n}$$

$$\therefore \frac{1}{2} \cdot \frac{f''(x_n)}{f'(x_n)} \cdot e_n^2 \cong (x_{n+1} - x_r)$$

$\therefore e_{n+1} = |x_r - x_{n+1}|$

$= |x_{n+1} - x_r|$

$$e_{n+1} \cong \frac{1}{2} \cdot \frac{f''(x_n)}{f'(x_n)} \cdot e_n^2.$$

$\therefore$ So it posses a second order convergence.

problem

Find the root of the equation

$x^3 - 2x - 5 = 0$ use newton raphson method.

A. Here $f(x) = x^3 - 2x - 5$

$\therefore f'(x) = 3x^2 - 2$

$\therefore x_{n+1} = x_n - \dfrac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$

by choosing $x_0 = 2$, we have

$f(x_0) = -1$, and $f'(x_0) = 10$.

putting n=0 we have

$x_1 = 2 - (-1/10) = 2.1$

$\therefore f(x_1) = (2.1)^3 - 2(2.1) - 5 = 0.061$

$f'(x_1) = 3.(2.1)^2 - 2 = 11.23$

$\therefore x_2 = 2.1 - \dfrac{0.061}{11.23} = 2.094568.$

$\therefore x_2 = 2.094568.$

problem Find a root of eqn using Newton raphson method, for $x^3 - 8x - 4 = 0$.

Ans 3.0514.

The Secant Method

~~Let~~ It is the method, like Regula falsi method. In regula falsi method we need two initial estimates but they need to bracket the root but in this method we need not bracket the root by ~~a~~ two initial estimates.

~~The~~ if two initial points are $(x_0, f(x_0))$ and $(x_1, f(x_1))$ then the equation of line pass through this points is,

$$\frac{y - y_1}{x - x_1} = \frac{y - y_0}{x - x_0}$$

say $x = x_2$ and, $y = 0$ [in $y = 0$ we have the value x_2, which is the solution of given equation $f(x) = 0$]

$$\therefore \frac{0 - y_1}{x_2 - x_1} = \frac{0 - y_0}{x_2 - x_0}$$

$$\text{or, } \frac{\cancel{-}y_1}{x_2 - x_1} = \frac{\cancel{-}y_0}{x_2 - x_0}$$

$$\text{or } (x_2 - x_0)\, y_1 = (x_2 - x_1)\, y_0$$

$$\text{or, } x_2 y_1 - x_0 y_1 = x_2 y_0 - x_1 y_0$$

$$\text{or, } x_2 y_1 - x_2 y_0 = x_0 y_1 - x_1 y_0$$

$$\text{or } x_2 (y_1 - y_0) = x_0 y_1 - x_1 y_0$$

$$x_2 = \frac{x_0 y_1 - x_1 y_0}{(y_1 - y_0)}$$

$$\therefore x_2 = \frac{f(x_1) \cdot x_0 - f(x_0) \cdot x_1 \cancel{- f(x_1) \cdot x_1 + f(x_1)}}{(f(x_1) - f(x_0))}$$

$$\text{or } x_2 = \frac{f(x_1) \cdot x_0 - f(x_0) \cdot x_1 + f(x_1) \cdot x_1 - f(x_1) \cdot x_1}{(f(x_1) - f(x_0))}$$

$$= \frac{f(x_1) \cdot (x_0 - x_1) + x_1 (f(x_1) - f(x_0))}{(f(x_1) - f(x_0))}$$

$$= \frac{x_1 (f(x_1) - f(x_0))}{(f(x_1) - f(x_0))} - \frac{(x_1 - x_0) \cdot f(x_1)}{f(x_1) - f(x_0)}$$

Therefore $$x_2 = x_1 - \frac{f(x_1)\cdot(x_1 - x_0)}{f(x_1) - f(x_0)}$$

Similarly we can obtain
$$x_{i+1} = x_i - \frac{f(x_i)\cdot(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

It is the linear interpolation polynomial in which interpolating points are $\{x_{i-1}, f(x_{i-1})\}$, $\{x_i, f(x_i)\}$ called secant formula.

This process is continued till the desired level of accuracy we can obtain.

__Algorithm__

Here root depends on two previous approximations.

1) Take two initial guess $\{x_0, f(x_0)\}$ and $\{x_1, f(x_1)\}$ as roots.

~~2) set $f_1 = f(x_1)$ and $f_2 = f(x_2)$.~~

2) Compute $x_2 = x_1 - \dfrac{f(x_1)\cdot(x_1 - x_0)}{f(x_1) - f(x_0)}$

3) if $\left(\left|\dfrac{x_1 - x_2}{x_1}\right| > \text{error}\right)$

then

4) set $x_0 = x_1$, and ~~x_0~~ $x_1 = x_2$

and goto ~~step~~ step 2,

otherwise

5) set root $= x_2$, print the result and exit.

6. stop.

__Problem__ Estimate the root of Eqn

$$x^2 - 4x - 10 = 0$$

use secant method, take initial guess $x_0 = 4$, $x_1 = 2$

Ans given $x_0 = 4$ and $x_1 = 2$.

$\therefore$ $f(x_0) = f(4) = -10$.

and $f(x_1) = f(2) = -14$.

So this guess do not brackets root.

now. from the formula

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$\therefore x_2 = 2 - \frac{(-14)(2-4)}{(-14)-(-10)}$$

$$= 2 - \frac{(-14)(-2)}{(-14)+10}$$

$$= 2 - \frac{28}{-4} = 2 + \cancel{28}^{7}/\cancel{4} = 2+7 = 9$$

$\therefore x_2 = 9$, $x_1 = 2$, $x_0 = 4$.