$\therefore x_2 = \frac{22}{5} - \frac{4}{5} \cdot x_3$

$= \frac{22}{5} - \frac{4}{5} \cdot 3 = \frac{22}{5} - \frac{12}{5} = \frac{22-12}{5} = \cancel{\frac{10}{5}} = 2$

$\therefore x_2 = 2.$

now, $x_1 + \frac{2}{3} \cdot x_2 + \frac{1}{3} \cdot x_3 = \frac{10}{3}$

or $x_1 + \frac{2}{3} \cdot 2 + \frac{1}{\cancel{3}} \cdot \cancel{3} = \frac{10}{3}$

or $x_1 + \frac{4}{3} + 1 = \frac{10}{3}$

or $x_1 = \frac{10}{3} - \frac{4}{3} - 1$

or $x_1 = \frac{10-4}{3} - 1$

or $x_1 = \frac{\cancel{6}^{2}}{\cancel{3}} - 1$

or $x_1 = 2 - 1 = 1$

$\therefore x_1 = 1$

now $X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$. Ans

Problem

Solve the system with

LU decomposition Crout's method.

$x_1 + x_2 - 2x_3 = 3$

$4x_1 - 2x_2 + x_3 = 5$

$3x_1 - x_2 + 3x_3 = 8.$

Gauss Seidal Iteration method

Direct methods have problems when the system will be large and most of the coefficients are '0'. They require a large no of floating point operations. For this it is not only become time consuming but also severely affect on accuracy of the solution due to round off errors. An ill-conditioned systems can be solved by iterative methods without facing the problem of round off errors.

In this method we start from an approximation to the true solution and, if convergent derive a sequence of closer approximations. Here the cycle of computation is being repeated till the desired accuracy is obtained. So, in a direct method the amount of computation is fixed while in case of iterative method the amount of computation depends on the accuracy required.

Let the system be given by

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$- \quad - \quad - \quad - \quad - \quad - \quad -$$
$$- \quad - \quad - \quad - \quad - \quad - \quad -$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n$$

In this system diagonal elements should not be vanished, i.e a_{ii}, if such case occurs equations should be rearranged, so that the condition satisfies.

next the system should be re-written as

$$x_1 = \left[b_1 - \{a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n\}\right] / a_{11}$$

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n\right]$$

similarly $$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n\right]$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2 - a_{34}x_4 - \cdots - a_{3n}x_n\right]$$

$$- \quad - \quad - \quad - \quad - \quad - \quad -$$

$$x_n = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - a_{n3}x_3 - \cdots - a_{n,n-1}x_{n-1}\right]$$

Let $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_n^{(1)}$ be the first approximation for the values $x_1, x_2, \ldots x_n$.

So, Second Approximation can be written as.

$$x_1^{(2)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)} - \cdots - a_{1n}x_n^{(1)}\right]$$

$$x_2^{(2)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(1)} - \cdots - a_{2n}x_n^{(1)}\right]$$

— — — — — — — — —

$$x_n^{(2)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(1)} - a_{n2}x_2^{(1)} \cdots - a_{n,n-1}x_{n-1}^{(1)}\right]$$

Similarly we can write

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \cdots - a_{1n}x_n^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - \cdots - a_{2n}x_n^{(k)}\right]$$

— — — — — — — — —

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)} - \cdots - a_{n,n-1}x_{n-1}^{(k)}\right]$$

— This computations steps is called Gauss-Jacobi-iteration method.

The general computational formula is

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - a_{i1}x_1^{(k)} - a_{i2}x_2^{(k)} - a_{i3}x_3^{(k)} - \cdots - a_{i,(i-1)}x_{i-1}^{(k)} - a_{i,(i+1)}x_{i+1}^{(k)} - \cdots - a_n x_n^{(k)}\right]$$

Here the Gauss-Seidal method is improved version of this Gauss-Jacobi method, here for quick convergence to the solution we use the currently obtained x_i values in computation of x_{i+1}, rather than obtaining current x_i value using all old x_i values.

Here the general formula is given by

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - a_{i1}x_1^{(k+1)} - a_{i2}x_2^{(k+1)} - a_{i3}x_3^{(k+1)} - \cdots - a_{i,i-1}x_{i-1}^{(k+1)} - a_{i,i+1}x_{i+1}^{(k)} - \cdots - a_{i,n}x_n^{(k)}\right].$$

Algorithm

1. Take input n, a_{ij}, and b_i values.
2. set $x_i = b_i/a_{ii}$ for $i=1$ to n [may be taken as $x_i=1$ for all i] initiallize the solution.
3. ~~set k=0~~
4. For $i=1$ to n do, $i=i+1$
5. Set $sum = b_i$
6. For $j=1$ to n $(j \neq i)$
7. set $sum = sum - a_{ij}x_j$
8. endfor.
9. set $d = sum/a_{ii}$
10. if $\left(\left|\frac{d-x_i}{d}\right| < \text{error}\right)$ then
11. set $x_i = d$, ~~and exit~~
12. ~~endif set exit~~ write results as x_i's, and exit().
13. endif.
14. ~~goto step~~ endfor.
15. goto step 4.

Problem:-

$$2x_1 + x_2 + x_3 = 5$$
$$3x_1 + 5x_2 + 2x_3 = 15$$
$$2x_1 + x_2 + 4x_3 = 8$$

Ans. $x_1^{(1)} = (5 - x_2^0 - x_3^0)/2.$

$x_2^{(1)} = (15 - 3x_1^{(1)} - 2x_3^0)/5$

and $x_3^{(1)} = (8 - 2x_1^{(1)} - x_2^{(1)})/4$

let $x_1^0 = 0$, $x_2^0 = 0$, $x_3^0 = 0$ then

After 1st iteration we have

$\cancel{x_1^{(1)} = (5-0-0)/2 = 5/2 = 2.5}$

$\cancel{x_2^{(1)} = (15-3.0-2.0)/5 = 15/5 = 3.0}$

$\cancel{x_3^{(1)} = (8-2.0-0)/4 = 8/4 = 2.0}$

$\cancel{\therefore x_1^{(1)} = 2.5,\ x_2^{(1)} = 3.0,\ x_3^{(1)} = 2.0.}$

~~now~~

$x_1^{(1)} = (5-0-0)/2 = 5/2 = 2.5 \quad \therefore x_1^{(1)} = 2.5$

$x_2^{(1)} = (15 - 3x_1^{(1)} - 2x_3^{(0)})/5$

$= (15 - 3.2.5 - 2.0)/5$

$= (15 - 7.5 - 0)/5$

$= 7.5/5 = 1.5$

$\therefore x_2^{(1)} = 1.5.$

now $x_3^{(1)} = (8 - 2x_1^{(1)} - x_2^{(1)})/4$

$= (8 - 2.2.5 - 1.5)/4$

$= (8 - 5 - 1.5)/4$

$= 1.5/4 = 0.375$

$\therefore x_3^{(1)} = 0.375.$

in iteration (2) we have

$x_1^{(2)} = (5 - x_2^{(1)} - x_3^{(1)})/2$

$= (5 - 1.5 - 0.375)/2$

$= (3.5 - 0.375)/2$

$= (3.125)/2$

$\therefore x_1^{(2)} = 1.5625$

now, $x_2^{(2)} = (15 - 3x_1^{(2)} - 2x_3^{(1)})/5$

$\therefore x_2^{(2)} = (15 - 3 \cdot 1.5625 - 2 \cdot 0.375)/5$

$= (15 - 4.6875 - 0.75)/5$ 0.375

$= (14.2500 - 4.6875)/5$

$= 9.5625/5$

$= 1.9125$

$\therefore x_2^{(2)} = 1.9125$ $2.8750 - 0.0875 = 2.9625$ (approx.)

and $x_3^{(2)} = (8 - 2x_1^{(2)} - x_2^{(2)})/4$

$= (8 - 2 \cdot 1.5625 - 1.9125)/4$

$= (8 - 3.125 - 1.9125)/4$

$= (4.8750 - 1.9125)/4$

$= 2.9625/4$

$= 0.740625$

$= 0.740625$

upto second iteration the value obtained is

$x_1^{(2)} = 1.5625$, $x_2^{(2)} = 1.9125$, $x_3^{(2)} = 0.740625$

~~Sufficient condition for convergence~~

Problems>

$$x_1 + x_2 + 4x_3 = 9$$
$$5x_1 - 3x_2 + 2x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$

Ans
After 3rd iteration
$x_1 = 3.00$, $x_2 = 2.00$
$x_3 = 1.00$

Sufficient condition for convergence

It is an iterative method and the method is based on finding successive better approximation of values of unknowns of the system of equations. The convergence of this iteration method is depends on the sufficient condition that the system should be diagonally dominant.

Let us a system of simultaneous equations are

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n$$

– The system will be ~~dig~~ diagonally dominant if $$|a_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \quad i=1 \text{ to } n.$$

if the system is not in diagonally dominant pattern, then we can re-arrange these equation and try to make it diagonally dominant.

Suppose let the set of equations given below.

$$x_1 + x_2 + 4x_3 = 9.$$
$$8x_1 - 3x_2 + 2x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$

Then ~~these~~ this is not in diagonally dominant format. then we re-arrange these equations as.

$$\left.\begin{aligned} 8x_1 - 3x_2 + 2x_3 &= 20 \\ 4x_1 + 11x_2 - x_3 &= 33 \\ x_1 + x_2 + 4x_3 &= 9 \end{aligned}\right\} \rightarrow \text{This is diagonally dominant in nature}$$