$$f(x) = y_n + u \cdot \nabla y_n + \frac{u(u+1)}{2!} \nabla^2 y_n + \frac{u(u+1)(u+2)}{3!} \nabla^3 y_n + \cdots + \frac{u(u+1)(u+2) \cdots (u+n-1)}{n!} \nabla^n y_n$$

$$\therefore f(21) = 15.4 + (0.2 \times 7.2) + \left\{ \frac{0.2(0.2+1)}{2} \times 1.4 \right\} + \frac{(0.2)(0.2+1)(0.2+2)}{6} \times 0.1$$

or $f(21) = 15.40 + 1.44 + (0.28 \times 1.2) + (0.02 \times 1.2 \times 2.2)$

or $f(21) = 16.840 + 0.336 + (0.02 \times 2.64)$

or $f(21) = 17.1760 + 0.0528$

$\therefore f(21) = 17.2288$.

Extrapolation.

From above problem we have seen that we determined the estimated value of $f(x)$, in correspondance of x, when $x = 21$, i.e beyond the limit of 'x_0 to x_n', we want to determine the corresponding value of $f(x)$ then this process is called extrapolation.

Newtons divided difference formula:

The main disadvantage of Lagrange interpolation formula is, if we want to add some points we have to recompute the whole procedure instead of adding a few terms with existing result. So, to overcome this problem we deduce an important formula called "Newton's General interpolation formula based on Divided difference technique.

Let $y = f(x)$ be known for $(n+1)$ distinct points (x_i, y_i) where $(i <= 0 \,\&\, i <= n)$, and this points are not necessarily equispaced.

Let a newtons polynomial be $p(x)$ of order 'n'

$$\therefore p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - \cdots$$

Now we know that, (x_0, f_0), (x_1, f_1), (x_2, f_2) — (x_n, f_n) points are known.

then

$$P(x_0) = a_0 = f_0$$

$$P(x_1) = a_0 + a_1(x_1 - x_0) = f_1$$

$$\therefore f_0 + a_1(x_1 - x_0) = f_1$$

$$\text{or } a_1(x_1 - x_0) = (f_1 - f_0)$$

$$\therefore a_1 = \frac{(f_1 - f_0)}{(x_1 - x_0)}$$

Again. $P(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f_2$

$$\therefore a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f_2$$

$$\text{or. } f_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f_2$$

$$\text{or. } (x_2 - x_0)\{a_1 + a_2(x_2 - x_1)\} = f_2 - f_0$$

$$\text{or } (x_2 - x_0)\left\{\frac{(f_1 - f_0)}{(x_1 - x_0)} + a_2(x_2 - x_1)\right\} = (f_2 - f_0)$$

$$\text{or } \frac{(f_1 - f_0)}{(x_1 - x_0)} + a_2(x_2 - x_1) = \frac{(f_2 - f_0)}{(x_2 - x_0)}$$

$$\text{or. } a_2(x_2 - x_1) = \frac{(f_2 - f_0)}{(x_2 - x_0)} - \frac{(f_1 - f_0)}{(x_1 - x_0)}$$

$$\text{or } a_2 \cdot (x_2 - x_1) = \frac{(f_2 - f_0)(x_1 - x_0) - (f_1 - f_0)(x_2 - x_0)}{(x_2 - x_0)(x_1 - x_0)}$$

$$= \frac{f_2 x_1 - f_2 x_0 - f_0 x_1 + \cancel{f_0 x_0} - f_1 x_2 + f_1 x_0 + f_0 x_2 - \cancel{f_0 x_0}}{(x_2 - x_0)(x_1 - x_0)}$$

or.

$$\therefore\ a_2(x_2-x_0) = \frac{f_2 x_1 - f_2 x_0 - f_0 x_1 + f_0 x_2 + f_1 x_0 - f_1 x_2}{(x_2-x_1)(x_1-x_0)}$$

$$= \frac{f_2(x_1-x_0) - f_1(x_1-x_0) + f_1 x_1 - f_1 x_0 - f_0 x_1 + f_0 x_2 - f_0 x_1 - f_1 x_2}{(x_2-x_1)(x_1-x_0)}$$

$$= \frac{(f_2-f_1)(x_1-x_0) - f_1(x_2-x_1) + f_0 x_2 - f_0 x_1 - f_1 x_0 + f_0 x_1}{(x_2-x_1)(x_1-x_0)}$$

$$(f_2-f_1)(x_1-x_0) - f_1(x_2-x_1) + f_0(x_2-x_1)$$

$$\cancel{f_2(x_1-x_0) - f_0 x_1 + f_0 x_2}$$

$$=$$

$$= \frac{f_2(x_1-x_0) + f_0(x_2-x_1) + f_1 x_0 - f_1 x_2}{(x_2-x_1)(x_1-x_0)}$$

$$= \frac{f_2(x_1-x_0) - f_1 x_1 + f_1 x_1 + f_1 x_0 - f_1 x_2 + f_0(x_2-x_1)}{(x_2-x_1)(x_1-x_0)}$$

$$= \frac{f_2(x_1-x_0) - f_1(x_1-x_0) - f_1(x_2-x_1) + f_0(x_2-x_1)}{(x_2-x_1)(x_1-x_0)}$$

$$= \frac{(f_2-f_1)(x_1-x_0) - (f_1-f_0)(x_2-x_1)}{(x_2-x_1)(x_1-x_0)}$$

$$= \left\{\frac{(f_2-f_1)}{(x_2-x_1)} - \frac{(f_1-f_0)}{(x_1-x_0)}\right\} \quad \therefore\ a_2 = \frac{\frac{(f_2-f_1)}{(x_2-x_1)} - \frac{(f_1-f_0)}{(x_1-x_0)}}{(x_2-x_0)}.$$

Now the notation is

$$a_0 = P(x_0) = f_0$$

$\therefore P(x_1) = f_1$

$$\therefore a_1 = \frac{f_1 - f_0}{x_1 - x_0} = f[x_0, x_1]$$

$$a_2 = \frac{\frac{(f_2 - f_1)}{(x_2 - x_1)} - \frac{(f_1 - f_0)}{(x_1 - x_0)}}{(x_2 - x_0)}$$

$$= \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$$

$$= f[x_0, x_1, x_2]$$

Similarly

$$a_3 = f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{(x_3 - x_0)}.$$

$$\therefore a_n = f[x_0, x_1, \ldots x_n] = \frac{f[x_1, x_2, \ldots x_n] - f[x_0, x_1, \ldots x_{n-1}]}{(x_n - x_0)}.$$

therefore $P(x)$ can be written as.

$$P(x) = f[x_0] + f[x_0, x_1].(x - x_0) + f[x_0, x_1, x_2].(x - x_0)(x - x_1)$$
$$+ \ldots + f[x_0, x_1, x_2, \ldots x_n](x - x_0)(x - x_1) \ldots (x - x_{n-1}).$$

$$= \sum_{i=0}^{n} f[x_0, x_1, \ldots, x_i] . \prod_{j=0}^{i-1} (x - x_j) \quad \text{------} \quad (1)$$

Now, we know that

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

if $x_1 = x_0 + h$ and $h \to 0$. then.

$$\lim_{h\to 0} f[x_0, x_1] = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$$

if $f(x)$ is differentiable.

again

$$\lim_{h\to 0} f[x_0, x_1, x_2] = \lim_{h\to 0} \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}.$$

$$= \frac{\lim_{h\to 0} f[x_1, x_2] - \lim_{h\to 0} f[x_0, x_1]}{2h}$$

$$= \lim_{h\to 0} \frac{\lim_{h\to 0} \frac{y_2 - y_1}{x_2 - x_1} - \lim_{h\to 0} \frac{y_1 - y_0}{x_1 - x_0}}{2h}$$

$$= \lim_{h\to 0} \frac{\lim_{h\to 0} \frac{f(x_1+h) - f(x_1)}{x_1 + h - x_1} - \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}}{2h}$$

$$= \lim_{h\to 0} \frac{f'(x_1) - f'(x_0)}{2h}$$

$$= \lim_{h\to 0} \frac{1}{2!} \frac{f'(x_0+h) - f'(x_0)}{h}$$

$$= \frac{1}{2!} \cdot f''(x_0)$$

Similarly $\lim_{h\to 0} f[x_0, x_1, x_2, x_3] = \frac{1}{3!} f'''(x_0)$

and $\lim_{h\to 0} f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(x_0)$.

Therefore for very small 'h' we can write

$$P(x) = f(x_0) + f'(x_0)\cdot(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)(x - x_1) + \frac{f'''(x_0)}{3!}(x - x_0)(x - x_1)(x - x_2) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \quad \text{---- (11)}$$

Again another form is,

$$f[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}.$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$$

$$= \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{(x_2 - x_0)}$$

$$= \frac{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}}{2h.} = \frac{\Delta y_1 - \Delta y_0}{2h^2} = \frac{\Delta^2 y_0}{2! h^2}$$

$$f[x_0, x_1, x_2, x_3) = \frac{\Delta^3 y_0}{3! \, h^3}$$

Similaly $f[x_0, x_1, x_2, \ldots x_n] = \frac{\Delta^n y_0}{n! \, h^n}$

So $P(x)$ can be again written as,

$$P(x) = f(x_0) + (x - x_0) \cdot \frac{\Delta y_0}{h} + (x - x_0)(x - x_1) \cdot \frac{\Delta^2 y_0}{2! h^2}$$
$$+ (x - x_0)(x - x_1)(x - x_2) \cdot \frac{\Delta^3 y_0}{3! \, h^3} +$$
$$\ldots + (x - x_0)(x - x_1)(x - x_2) \ldots (x - x_{n-1}) \cdot \frac{\Delta^n y_0}{n! \, h^n}$$

(iii)

This i, ii, iii, → are called newtons general interpolation formula and specially formula no. (i) is known as formula for divided difference

Q. A set of data points given below:

x_i :	1	2	3	4	5
$f(x_i)$:	0	7	26	63	124

Find the divided difference table and hence find $f(1.5)$.

Ans

x_i	$f(x_i)$	first order divided diff	second order divided diff	third order divided diff	fourth order divided diff
1	0				
		7			
2	7		$\cancel{12}$ 6		
				$\cancel{8}$ 1	
		19			0.
3	26		$\cancel{18}$ 9		
				$\cancel{6}$ 1	
		37			
			$\cancel{24}$ 12		
4	63				
		61			
5	124				

Here $h=1$, $x_0=1$

$f[x_0] = 0$

$f[x_0, x_1] = \cancel{\frac{\Delta y_0}{h}} = \cancel{\frac{7}{1}} = 7$

$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$

$= \cancel{\frac{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}}{2h}} = \cancel{\frac{\Delta y_1 - \Delta y_0}{2h}} = \cancel{\frac{\Delta^2 y_0}{2 \cdot 1}} = \cancel{\frac{1}{2} \cdot \Delta^2 y_0} = \cancel{\frac{1}{2} \cdot 12 = 6}$

This operation is not needed for divided difference algorithm.

So similarly,

$f[x_0] = 0$, $f[x_0, x_1] = 7$, $f[x_0, x_1, x_2] = 6$

and $f[x_0, x_1, x_2, x_3] = 1$

lastly $f[x_0, x_1, x_2, x_3] = \cancel{\frac{\Delta^3 y_0}{3! h^3} = \frac{6}{6 \cdot 1^3} = \frac{6}{6} = 1}$.

therefore

$$P(x) = 0 + 7(x - x_0) + 6(x - x_0)(x - x_1) + 1 \cdot (x - x_0)(x - x_1)(x - x_2)$$

$$= 7(x-1) + 6(x-1)(x-2) + (x-1)(x-2)(x-3)$$

$$= (x-1)\{7 + 6(x-2) + (x-2)(x-3)\}$$

$$= (x-1)\{7 + (x-2)(x-3+6)\}$$

$$= (x-1)\{7 + (x-2)(x+3)\}$$

$$= (x-1)\{7 + x^2 + x - 6\}$$

$$= (x-1)(x^2+x+1)$$

$\therefore P(x) = x^3 - 1$ $\quad \therefore P(1.5) = (1.5)^3 - 1 = 3.375 - 1 = 2.375$

$\therefore P(1.5) = 2.375$ Ans

Q. Using the following table find $f(x)$ as a polynomial in x

x :	-1	0	3	6	7
$f(x)$:	3	-6	39	822	1611

→ Use newtons divided difference formula.

Ans

x	$f(x)$	first order	second order	third order	fourth order
-1	3				
		-9			
0	-6		6		
		15		5	
3	39		41		1
		261		13	
6	822		132		
		789			
7	1611				

therefore

$$P(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) + \cdots + f[x_0, x_1, x_2 \cdots x_n](x-x_0)(x-x_1) \cdots (x-x_{n-1})$$

$$= 3 + (-9).(x-(-1)) + 6(x-(-1))(x-0) + 5(x-(-1))(x-0)(x-3) + 1(x-(-1))(x-0)(x-3)(x-6)$$

$$= 3 - 9(x+1) + 6(x+1).x + 5(x+1).x.(x-3) + (x+1).x.(x-3)(x-6)$$

$$\cancel{= 3 - 9x - 9 + 6x^2 + 6x}$$

$$= 3 - 9(x+1) + 6(x+1).x + x(x+1)(x-3)\{x-6+5\}$$

$$= 3 - 9(x+1) + 6(x+1).x + x(x+1)(x-3).(x-1)$$

$$\cancel{= 3 - (x+1)\{9 - 6x\}}$$

$$= 3 - 9(x+1) + x(x+1)\{(x-3)(x-1) + 6\}$$

$$= 3 - 9(x+1) + x(x+1).(x^2 - 4x + 3 + 6)$$

$$= 3 - 9(x+1) + x(x+1)(x^2 - 4x + 9)$$

$= 3 - (x+1)\{9 - x(x^2-4x+9)\}$

$= 3 - (x+1)\{9 - x^3 + 4x^2 - 9x\}$

$= 3 + (x+1)\{x^3 - 4x^2 + 9x - 9\}$

$= 3 + x^4 - 4x^3 + 9x^2 - \cancel{9x} + x^3 - 4x^2 + \cancel{9x} - 9$

$= x^4 - 3x^3 + 5x^2 - 6$

$\therefore P(x) = x^4 - 3x^3 + 5x^2 - 6.$

So this is the desired polynomial using newtons divided difference formula.

<u>Algorithm for constructing a divided difference table</u>

$(n+1)$ data points (x_i, f_i) where $i=0, i<=n, i=i+1$

1) for $i=1$ to n do

2) for ($j=0$ to $(n-i)$ do.

3) Compute $f[x_j, x_{j+1}, \ldots, x_i] = (f[x_{j+1}, x_{j+2}, \ldots x_{i}]$

$- f[x_j, x_{j+1}, x_{j+2}, \ldots x_{i-1}]) / (x_{j+i} - x_j)$

4) enter into column i of the table.

5) end of step 2 loop

6) end of step 1 loop.

7) exit

determining estimating value corresponding to x.

1. Take input the key value as 'x'
2. Set sum $= f_0$,
3. for $i=1$ to n do
4. set $t=1$.
5. for ($j=1$ to i do.
6. $t = t * (x - x[j-1])$
7. endfor.
8. set sum $=$ sum $+ t * a[1][i]$ [where a is the divided difference storage array]
9. endfor.
10. print sum as estimated value
11. exit

Error in divided difference formula.

Error occured in newtons divided difference formula is identical to error occured in lagrange interpolation polynomial

$$E(x) = (x-x_0)(x-x_1) \cdots (x-x_n) \cdot \frac{f^{n+1}(\xi)}{(n+1)!}$$

Problems: