Curve fitting

Sometimes we need to estimate the value of dependent variable (y), corresponding to its independent variable (x). In this regard when a set of data points (xi, yi) is given and we have to determine the function f(x) y = f(x), which is approximate function and passes through the data points given. Here we can put the value of x and get the estimated value of y as f(x). Here the process of constructing f(x) from given data point is called curve fitting.

This ~~data~~ set of data points (x_i, y_i), i = 0, 1, ... ~~have two categories.~~ are of two types.

1) Set of values of well-defined functions:— which means the set of values follows a well defined function like trigonometric, logarithmic etc.

2) Set of values comes from experimental data like It may comes from relation between waterfall (rain) and production of crops, relation between voltage and rotation of fan in rpm etc.

For set of data type 1, we already have well defined functions but for type 2 there is no such function, in this case we try to fitt available and well defined curve to that data points and which curve is best suited for that points is taken for estimating the value of dependent variable (y) in correspondence of ~~indepen~~ given independent variable.

We have two approaches to fit curve to the given set of data points.

1) Interpolation , (ii) Regression.

① Interpolation:-

In this method, when we have given a set of data points, we try to fit a ~~polynomial available say $\phi(x)$~~ available polynomial say $\phi(x)$ to the data points first. Then the polynomial $\phi(x)$ is called interpolation polynomial which passes through each an every data points given. Using this approximate curve $\phi(x)$ we estimate the value of y_i corresponding to given x_i.

There are many methods available in interpolation. They are different in name due to using of different polynomials for fitting into data points.

They are
1) Lagrange Interpolation
2) Newton's interpolation
3) Newton - Gregory forward interpolation

etc. many such methods are there. But they are effective in different different cases.

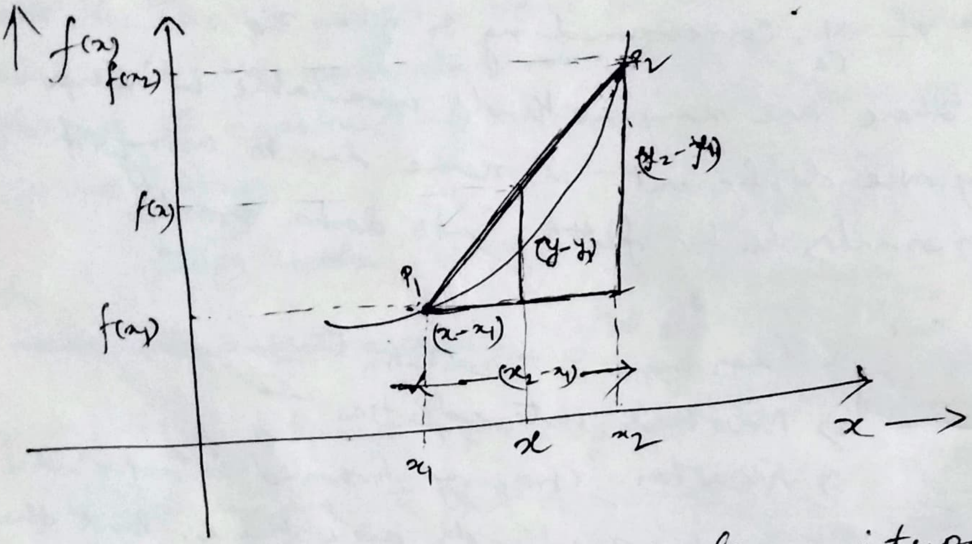
② Regression:-

In this case set of data points is given. But we are not try to fit such a curve which passes through each and every point given. Instead in this case we take any line or curve (standard polynomial) and updates that lines or curves parameters (coefficients) according to given data points, so that this line or curve can represent the general trend of the data and not necessary it has to pass all the ~~data~~ given data points.

Newton's forward Interpolation formula:-

Linear Interpolation

The simplest of all interpolation is linear interpolation formula. Here ~~only~~ we estimate the value of dependent variable using two points, say $\{x_1, f(x_1)\}$, and $\{x_2, f(x_2)\}$. The curve passing through this points is assumed to be a straight line.



Graphical representation of linear interpolation.

From the concept of similar traingle we get,

$$\frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(y - y_1)}{(x - x_1)}$$

or. $$(y - y_1) = (x - x_1) \cdot \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

or $$y = y_1 + (x - x_1) \cdot \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

$$\therefore \quad f(x) = f(x_1) + (x - x_1) \cdot \frac{f(x_2) - f(x_1)}{(x_2 - x_1)}$$

This formula is known as linear interpolation formula.

x	1	2	3	4	[illegible]
$f(x)$	1	1.4142	1.732	2	[illegible]

Determine the square root of 3.5, using linear interpolation formula.

Ans.

Here, the value of 3.5 is between 3 and 4.

Then say $x_1 = 3$, $f(x_1) = 1.732$, $x = 3.5$

$x_2 = 4$, $f(x_2) = 2$

Hence from linear interpolation formula, we get

$$f(3.5) = f(3) + (3.5 - 3) \cdot \frac{f(4) - f(3)}{(4 - 3)}$$

$$\text{or, } f(3.5) = 1.732 + \frac{1}{2} \cdot \frac{2 - 1.732}{1}$$

$$= 1.732 + \frac{1}{2} \cdot 0.268$$

$$= 1.732 + .134$$

$$\therefore f(3.5) = 1.866 \text{ Ans.}$$

Lagrange interpolation formula.

For $(n+1)$ distinct unequally spaced points. If we use ~~we use~~ Lagrange interpolation polynomial ~~through some~~ of the degree, $(n+1)$ points, then the polynomial ~~[illegible]~~ is of

Let the points be $(x_0, f_0), (x_1, f_1), \ldots (x_n, f_n)$ are given. And the ~~Lagrange interpolation polynomial~~ function say $f(x)$ be a continuous and differentiable $(n+1)$ times in the interval (a, b).

Now say another polynomial is $P(x)$.

for which $P(x_i) = y_i = f_i \; [y = f(x_i), \text{ for } i = 0, 1, 2, \ldots n]$

That is a interpolation function is such a function (curve) that passes through each and every data point which is given.

For linear interpolation say there are two points (x_0, y_0) and (x_1, y_1).

then

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{y - y_0}{x - x_0}$$

$$\Rightarrow \frac{(x - x_0)}{(x_1 - x_0)}(y_1 - y_0) = (y - y_0)$$

$$\Rightarrow \frac{x - x_0}{x_1 - x_0} y_1 - \frac{x - x_0}{x_1 - x_0} y_0 = y - y_0$$

$$P_1(x) = y = \frac{x - x_0}{x_1 - x_0} y_1 + y_0 - \frac{x - x_0}{x_1 - x_0} y_0$$

$$= \frac{x - x_0}{x_1 - x_0} y_1 + \left[1 - \frac{x - x_0}{x_1 - x_0}\right] y_0$$

$$= \frac{x - x_0}{x_1 - x_0} y_1 + \left[\frac{x_1 - x_0 - x + x_0}{x_1 - x_0}\right] y_0$$

$$= \frac{(x_1 - x)}{(x_1 - x_0)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

$$= \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

$$= l_0(x) y_0 + l_1(x) y_1$$

$$= \sum_{i=0}^{1} l_i(x) \cdot y_i$$

This the Lagrange polynomial of degree 1, passing through two points.

Now, the Lagrange interpolation polynomial of degree 2, is passing through 3 points (x_0, y_0), (x_1, y_1), (x_2, y_2).

Let us consider a second order polynomial

$$P_2(x) = b_1(x-x_0)(x-x_1) + b_2(x-x_1)(x-x_2) + b_3(x-x_2)(x-x_0)$$

if (x_0, y_0), (x_1, y_1) and (x_2, y_2) are interpolating points then.

$$P_2(x_0) = b_2(x_0-x_1)(x_0-x_2) = y_0$$

$$\therefore\ b_2 = \frac{y_0}{(x_0-x_1)(x_0-x_2)}$$

Similarly

$$P_2(x_1) = b_3(x_1-x_2)(x_1-x_0) = y_1$$

$$\therefore\ b_3 = \frac{y_1}{(x_1-x_2)(x_1-x_0)}$$

and $$b_1 = \frac{y_2}{(x_2-x_0)(x_2-x_1)}$$

By substituting this values we get

$$P_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2 + \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)} y_1$$

$$\therefore P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2.$$

$$= l_0(x)\cdot y_0 + l_1(x)\, y_1 + l_2(x)\, y_2$$

$$= \sum_{i=0}^{2} l_i(x)\cdot y_i$$

In general for $(n+1)$ points passing lagrange interpolation polynomial is of nth degree

and, $$P_n(x) = \sum_{i=0}^{n} l_i(x) \cdot y_i \quad \text{--- (i)}$$

where $$l_i(x) = \prod_{j=0, j\neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)} \quad \text{--- (ii)}$$

equation (ii) is called lagrange's basis polynomial.

and, equation (i) is called lagranges interpolation polynomial of degree 'n'.

and, $$l_i(x_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

Problem

x_i	0	1	2	3
$e^{x_i} - 1$	0	1.7183	6.3891	19.0855

Use the polynomial to estimate the value of $e^{1.5}$

Ans. As the four point is given, the polynomial should of degree $(n-1) = (4-1) = 3$.

now. lagrange interpolation polynomial is of order 3

$$P_3(x) = \sum_{i=0}^{3} l_i(x) \cdot y_i$$

$$= l_0(x) \cdot y_0 + l_1(x) \cdot y_1 + l_2(x) \cdot y_2 + l_3(x) \cdot y_3$$

here $y_0 = 0$

$$\therefore P_3(x) = l_1(x) \cdot y_1 + l_2(x) \cdot y_2 + l_3(x) \cdot y_3$$

now $l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$

$= \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{x(x-2)(x-3)}{2}$

$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)\cdot(x_2-x_3)}$

$= \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} = \frac{-x(x-1)(x-3)}{2}$

$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$

$= \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \frac{x(x-1)(x-2)}{6}$

$\therefore P_3(x) = \frac{x(x-2)(x-3)}{2} \cdot 1.7183 - \frac{x(x-1)(x-3)}{2} \cdot 6.3891 + \frac{x(x-1)(x-2)}{6} \cdot 19.0855$

$= \frac{5.0732x^3 - 6.3621x^2 + 11.5987x}{6}$

$= 0.8455x^3 - 1.0604x^2 + 1.9331x$

$\therefore P_3(1.5) = e^{1.5} - 1 = 3.3677$

$\therefore e^{1.5} = 1 + 3.3677 = 4.3677$

Algorithm for computing interpolation value using Lagrange interpolation polynomial.

(1). take the values of (x_i, y_i) where $i = 0, 1, 2, \ldots n$ (that is $(n+1)$ points). as input, where $y_i = f(x_i)$

(2) set Key = The value of x for which we want to calculate the value of corresponding P.f (interpolating value)

(2a) set P = 0

(3) Repeat step 4 to 8 till i<=n, incremented by 1, i starts with '1'.

~~(4) Set P = P + l.y~~

(4a) l = 1;

(5) Repeat step 6 to 7 till j <= n, 'j' value starts with '0', incremented by 1,

(6) if (i ≠ j)

set $l = l * \frac{(Key - x_j)}{(x_i - x_j)}$

(7) endif

(8) set $P = P + l.y_i$

(8a) end of step 3 loop.

(9) Print the value of P

(10) End.

Q. Find the form of interpolating function f(x), from the following table.

x :	0	1	3	4
y :	-12	0	12	24

Ans: As $y = f(x) = 0$, when $x = 1$,

then obviously $(x-1)$ is a factor of $f(x)$.

$$f(x) = (x-1) \cdot g(x).$$

$$\therefore g(x) = \frac{f(x)}{(x-1)}$$

Now we will try to find the form of $g(x)$.

x :	0	3	4
g(x) :	$\frac{-12}{(0-1)}$	$\frac{12}{(3-1)}$	$\frac{24}{(4-1)}$

$$\begin{array}{llll} & x_0 & x_1 & x_2 \\ \therefore\ x: & 0 & 3 & 4 \\ g(x): & 12 & 6 & 8 \end{array}$$

From lagrange interpolation polynomial we have.

$$p(x) = \sum_{i=0}^{2} g(x_i) \cdot l_i(x)$$

$$= l_0(x) \cdot g(x_0) + l_1(x) \cdot g(x_1) + l_2(x) \cdot g(x_2)$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot g(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot g(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot g(x_2)$$

$$= \frac{(x-3)(x-4)}{(0-3)(0-4)} \cdot 12 + \frac{(x-0)(x-4)}{(3-0)(3-4)} \cdot 6 + \frac{(x-0)(x-3)}{(4-0)(4-3)} \cdot 8$$

$$= \frac{12(x-3)(x-4)}{12} + \frac{6x(x-4)}{3 \cdot (-1)} + \frac{x(x-3)}{4 \cdot 1} \times 8$$

$$= (x-3)(x-4) - 2x(x-4) + 2x(x-3)$$

$$= (x-3)(x-4) + 2x(x-4-x+3)$$

~~$= x^2 - 7x + 12 + 4x^2 - 14x$~~

~~$= (5x^2 - 21x + 12)$~~

$$= x^2 - 7x + 12 + 2x$$

$$= (x^2 - 5x + 12)$$

$\therefore$ required interpolating polynomial

$$f(x) = (x-1)(x^2-5x+12).$$

$$= x^3 - 5x^2 + 12x - x^2 + 5x - 12$$

$$= x^3 - 6x^2 + 17x - 12$$

Advantage and disadvantage of lagrange's interpolation:

Advantages.

The main advantage of lagrange interpolation formula is it can be used for equispaced or unequispaced points given. And there is no restriction on 'x' → for which we are going to compute 'y = f(x)', — i.e x may lie at the begining end or middle of the interval. $[x_0 ; x_n]$.

Disadvantage:

For computation by this method the whole data comes into computation which so laborious. And if we add some extra point within my calculation then whole calculation should be perfomed from begining.

Error in Lagrange interpolation polynomial

$$R_{n+1}(x) = f(x) - L_n(x) = \frac{P_{n+1}(x)}{(n+1)!} f^{(n+1)}(\xi)$$

where $x_0 < \xi < x_n$
ξ → any value.

$L_n(x)$ ⇒ interpolation polynomial of nth degree

$f(x)$ ⇒ is the Known function which is continuously and derivable upto at least $(n+1)$th time.

$R_{n+1}(x)$ ⇒ Error obtained.

Actually for knowing the effectiveness of the lagranges interpolation polynomial, we starts with the set of points obtain from a well defined fuction which is continuous and derivable upto nth time.

Q. Find the lagrange interpolating polynomial of degree 2. Take the approximating function as $y = \log_e x$, then determine the error in calculation of $\log_e 2.7$.

	x_0	x_1	x_2
x :	2	2.5	3.0
$y = \log_e x = f(x)$:	0.69315	0.91629	1.09861

Ans. Here we have seen, that only three points are given, so we can have the lagrange interpolation of degree $(3-1) = 2$.

Therefore $P_2(x) = \sum_{i=0}^{2} f_i(x) \cdot l_i(x)$

Here $x = 2.7$

where $P_2(x)$ is interpolating polynomial.

$f_i \rightarrow$ values are given.

$l_i(x) \Rightarrow$ lagranges basis polynomial.

$\therefore l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(2.7-2.5)(2.7-3.0)}{(2-2.5)(2-3.0)}$

$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(2.7-2)(2.7-3.0)}{(2.5-2)(2.5-3.0)}$

$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(2.7-2)(2.7-2.5)}{(3.0-2)(3.0-2.5)}$

$\therefore l_0(x) = \frac{-.2 \times .3}{.5 \times 1} = \frac{-.06}{1/2} = -.06 \times 2 = -.12$

$l_1(x) = \frac{.7 \times .3}{.5 \times .5} = \frac{.21}{.25} = \frac{21}{25} = .84$

$l_2(x) = \frac{.7 \times .2}{1 \times .5} = \frac{.14}{.5} = \frac{14}{5} = 2.8$

$\therefore P_2(2.7) = \sum_{i=0}^{2} f_i \, l_i(x) = f_0 l_0 + f_1 l_1 + f_2 l_2$

$= (0.69315 \times .12) + (0.84 \times 0.91629)$

$+ (2.8 \times 1.09861) =$

Newtons forward Interpolation formula

Let $f(x)$ is a function that is known for $(n+1)$ distinct equispaced ~~points~~ arguments $x_0, x_1, x_2, x_3 \ldots x_r, \ldots x_n$

where $x_r = x_0 + rh$

and $x_n = x_0 + nh$ h is the length of each space

Now, $f(x_0) = y_0$, $f(x_1) = y_1$, $f(x_2) = y_2 \ldots$

$f(x_r) = y_r$, $\ldots$ $f(x_n) = y_n$

$$\therefore x_n - x_r = (x_0 + nh) - (x_0 + rh)$$
$$= x_0 + nh - x_0 - rh$$
$$= (n-r)h.$$

Now we take a polynomial $P_n(x)$ of degree 'n' which exists all points where $f(x)$ exists.

$$\therefore P_n(x_i) = f(x_i) = y_i \quad \& \text{ where } \{ i = 0, 1, 2, \ldots n \}$$

Let $P_n(x)$ is a polynomial of degree 'n'.

$$\therefore P_n(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)(x - x_1)$$
$$+ A_3(x - x_0)(x - x_1)(x - x_2) + \ldots$$
$$+ \ldots + A_n(x - x_0)(x - x_1)(x - x_2) \ldots (x - x_{n-1})$$

now, $P_n(x_0) = A_0 = y_0$

$\therefore A_0 = y_0$

$P_n(x_1) = A_0 + A_1(x_1 - x_0)$

or $y_1 = y_0 + A_1(x_1 - x_0)$ $\therefore A_1 = \dfrac{(y_1 - y_0)}{(x_1 - x_0)}$ $\therefore A_1 = \dfrac{\Delta y_0}{h}$

$P_n(x_2) = A_0 + A_1(x_2 - x_0) + A_2(x_2 - x_0)(x_2 - x_1) = y_2$

$\therefore y_0 + A_1 \cdot 2h + A_2(2h)(h) = y_2$

$\Rightarrow y_0 + 2h \cdot A_1 + 2h^2 \cdot A_2 = y_2$

$\Rightarrow y_0 + 2h \cdot \frac{\Delta y_0}{h} + 2h^2 A_2 = y_2$

$\Rightarrow y_0 + 2(y_1 - y_0) + 2h^2 A_2 = y_2$

$\Rightarrow y_0 + 2y_1 - 2y_0 + 2h^2 A_2 = y_2$

$\Rightarrow 2y_1 - y_0 + 2h^2 A_2 = y_2$

$\Rightarrow 2h^2 A_2 = y_2 - 2y_1 + y_0$

$\Rightarrow 2h^2 A_2 = \Delta^2 y_0$

$\therefore A_2 = \frac{\Delta^2 y_0}{2! h^2}$

Similarly we can have $A_n = \frac{\Delta^n y_0}{n! h^n}$

~~now let, $u = \frac{x - x_0}{h}$ $\therefore$ $x = x_0 + hu$~~

So by substituting the values of A we get the polynomial as

$$P_n(x) = y_0 + (x - x_0) \cdot \frac{\Delta y_0}{h} + (x - x_0)(x - x_1) \cdot \frac{\Delta^2 y_0}{2! h^2}$$
$$+ (x - x_0)(x - x_1)(x - x_2) \cdot \frac{\Delta^3 y_0}{3! h^3} + \cdots$$
$$+ (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1}) \frac{\Delta^n y_0}{n! h^n}$$

now let $u = \frac{x - x_0}{h}$ or $x = x_0 + hu$.

and also $x_r = x_0 + rh$.

$\therefore (x - x_r) = (u - r) \cdot h$

$\therefore P_n(x) = P_n(x_0 + hu)$

$$= y_0 + u \cdot \Delta y_0 + \frac{u(u-1)}{2!} \cdot \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0$$

$$+ \cdots + \frac{u(u-1)(u-2) \cdots (u-n+1)}{n!} \Delta^n y_0$$

This formula is known as newtons forward interpolation formula.

Error estimation:

The error committed in replacing $f(x)$ by $P_n(x)$ in Newton's forward ~~form~~ Interpolation formula as.

$$R_{n+1}(x) = \frac{(x-x_0)(x-x_1)(x-x_2) \cdots (x-x_{n-1})(x-x_n)}{(n+1)!} f^{n+1}(\xi)$$

where ξ is such that $(x_0 < \xi < x_n)$

~~$\min(x, x_0, x_n) < \xi < \max(x, x_0, x_n)$~~

let $u = \frac{x - x_0}{h}$ and $x_r = x_0 + rh$

$\therefore (u - r)h = (x - x_r)$

$$\therefore R_{n+1}(x) = \frac{u(u-1)(u-2) \cdots (u-n+1)(u-n)}{(n+1)!} \times h^{n+1} f^{n+1}(\xi)$$

$$\therefore R_{n+1}(x) = \frac{u(u-1)(u-2) \cdots (u-n+1)(u-n)}{(n+1)!} \Delta^{n+1} y_0$$

Algorithm to implement newtons forward difference table

1. take all $y = f(x)$ values in an array a[ ][ ] when 'n' is the total no of elements.
2. set $i = 1$;
3. Repeat step 4 to 9 until $j <= n-1$, j starts from 2, incremented by 1.
4. Repeat step 5 & 6 until $k <= n-i$, k starts from 1, incremented by 1.
5. set a[k][j] = a[k+1][j-1] − a[k][j-1],
6. ~~print a[k][j], and go to next line~~
6. end of step 4 loop.
7. ~~print "double space"~~,
7. set $i = i+1$;
8. End of step 3 loop.
9. print a[ ][ ]
10. exit.

Algorithm for obtaining estimated value using forward interpolation formula, against x value given.

1. Get the all x_i values.
2. Obtained equal interval width $h = x_{i+1} - x_i$
3. set x_0 is the first value of all x_i's.
4. take the value p, against which we have estimate the corresponding f(p) value.
5. set $u = \frac{p - x_0}{h}$
6. Set sum = y_0 = $f(x_0)$
6(a). use the above algorithm. → 6(b) → t=1
7. Repeat step 8 to 10 until $j <= n-1$, j starts with 2, increments by 1.
8. ~~set sum = sum + u × a[1][j] / factorial (j−1) incremented by 1~~,
8. set t = t × u
9. set g = t × a[1][j] / factorial (j−1);
10. set $u = u - 1$;
11. end of step 7 loop.
12. print the value of 'sum'.
13. exit

Q. Given the following table

x :	0	5	10	15	20
$f(x)$:	1.0	1.6	3.8	8.2	15.4

Calculate $f(4)$ using newtons forward interpolation formula.

Soln: The difference table is

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1.0	0.6	1.6	0.6	0
5	1.6	2.2	2.2	0.6	
10	3.8	4.4	2.8		
15	8.2	7.2			
20	15.4				

Now, by using the newtons forward interpolation formula,

$$P_n(x) = y_0 + u.\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

here $\Rightarrow u = \frac{x - x_0}{h}$

when $x_0 = 0, \quad x = 4, \quad h = 5$

$\therefore u = \frac{4-0}{5} = 0.8$

$y_0 = 1.0$

$\therefore P_n(4) = 1.0 + (0.8 \times 0.6) + \frac{0.8(0.8-1)}{2!}.1.6 + \frac{0.8(0.8-1)(0.8-2)}{3!} \times (0.6)$

$= 1.0 + 0.48 + \frac{0.8 \times (-0.2)}{2}.1.6 + \frac{0.8 \times (-0.2) \times (-1.2)}{6} \times (0.6)$

$= 1.48 - 0.128 + 0.8 \times .24$

$= 1.480 - 0.128 + 0.192 = 1.352 + 0.192 = 1.544.$

1.3712

$\therefore P_n(4) = \sout{1.544}\ 1.3712$.

Newtons backward Interpolation formula.

Let a function $f(x)$ is known in $(n+1)$ equispaced points. where the points are, $x_0, x_1, x_2 \ldots x_r, \ldots x_{n-1}, x_n$. here '$h$' is the width between consequitive x values.

now. $x_r = x_0 + rh$

Again $f(x_0) = y_0$, $f(x_1) = y_1$, $f(x_2) = y_2$; ...
... $f(x_r) = y_r$ ...... $f(x_n) = y_n$

~~$x_{n-r} = \{x_0 + (n-r)h\} - \{x_0 + rh\} = x_0 + nh - rh = \ldots = \ldots h$~~

Now, assume

$$P(x) = B_0 + B_1(x - x_n) + B_2(x - x_n)(x - x_{n-1}) + \ldots + B_n(x - x_n)(x - x_{n-1})(x - x_{n-2}) \ldots (x - x_2)(x - x_1).$$

now, $P(x_n) = B_0 = y_n$.

$$P(x_{n-1}) = B_0 + B_1(x_{n-1} - x_n)$$
$$= y_n - B_1(x_n - x_{n-1})$$
$$= y_n - B_1 . h$$

or. $y_{n-1} = y_n - B_1 . h$

or. $B_1 . h = y_n - y_{n-1}$

or. $B_1 = \frac{\nabla y_n}{h}$,

now. $P(x_{n-2}) = B_0 + B_1(x_{n-2} - x_n) + B_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$

or. $y_{n-2} = y_n + B_1 . \underset{(-1)}{2h} + B_2 . . 2h^2$

$\therefore y_{n-2} - y_n + B_1 . 2h = B_2 . 2h^2$

$\therefore y_{n-2} - y_n + 2(B_1 h) = B_2 . 2h^2$

$\therefore y_{n-2} - y_n + 2(y_n - y_{n-1}) = B_2 . 2h^2$

$\therefore y_{n-2} - y_n + 2y_n - 2y_{n-1} = B_2 . 2h^2$

$\therefore y_n - 2y_{n-1} + y_{n-2} = B_2 . 2h^2$

$\therefore (y_n - y_{n-1}) - (y_{n-1} - y_{n-2}) = B_2 . 2h^2$

$\therefore \nabla y_n - \nabla y_{n-1} = B_2 . 2h^2$

$\therefore \nabla^2 y_n = B_2 . 2h^2$

$\therefore B_2 = \dfrac{\nabla^2 y_n}{2!\, h^2}$

Similarly $B_n = \dfrac{\nabla^n y_n}{n!\, h^n}$

now, $P(x) = y_n + (x - x_n) . \dfrac{\nabla y_n}{h} + (x - x_n)(x - x_{n-1}) . \dfrac{\nabla^2 y_n}{2!\, h^2}$

$+ (x - x_n)(x - x_{n-1})(x - x_{n-2}) \dfrac{\nabla^3 y_n}{3!\, h^3}$

$+ - - - -$

$+ (x - x_n)(x - x_{n-1})(x - x_{n-2}) . - -$

$- - - (x - x_2)(x - x_1)(x - x_0) .$

now. let $u = \dfrac{x - x_n}{h}$ $\therefore x = x_n + u . h$

and. $x_{n-r} = x_n - rh$

$\therefore x - x_{n-r} = u . h + rh + x_n - x_n$

$\therefore (x - x_{n-r}) = (u + r) h .$

$$\therefore P(x) = y_n + u\cdot\nabla y_n + u(u+1)\cdot\frac{\nabla^2 y_n}{2!} + u(u+1)(u+2)\cdot\frac{\nabla^3 y_n}{3!}$$

$$+ \cdots + u(u+1)(u+2)\cdots(u+n)\cdot\frac{\nabla^n y_n}{n!}$$

This is the newtons backward interpolation polynomial.

Error estimation:

The error in newtons backward interpolation formula

$$R_{n+1}(x) = \frac{u(u+1)(u+2)\cdots(u+n)}{(n+1)!}\, h^{n+1} f^{n+1}(\xi) \quad \text{where } (x_0<\xi<x_n)$$

$$= \frac{u(u+1)(u+2)\cdots(u+n)}{(n+1)!}\cdot \nabla^{n+1} y_{n+1}$$

~~$\therefore |R_{n+1}(x)| = |B_n(u)\ldots|$~~

This newton's backward formula we use when we have to determine the value of dependent variable in correspondance of independent variable which is closer to the end-points.

problem:-

It is given that,

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

Find $f(7.5)$.

Ans

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	1	7	12	6	0	
2	8	19	18	6	0	
3	27	37	24	6	0	
4	64	61	30	6	0	
5	125	91	36	6		
6	216	127	42			
7	343	169				
8	512					

Fig (1)

though it is a forward difference table we can use last diagonal element for backward difference

$$u = \frac{x - x_n}{h}$$

$$= \frac{7.5 - 8}{1} = 0.5$$

$\therefore u = -0.5$

$$P(x) = 512 + (-0.5).169 + \frac{(-0.5)(-0.5+1)}{2!}.42 + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}.6$$

$$= 512 - 84.5 + \frac{(-0.5)(0.5)}{2}.42 + \frac{(-0.5)(0.5)(1.5)}{6}.6$$

$$= 512 - 84.5 - 21.(.25) - .25 \times 1.5$$

$$= 512 - 84.500 - 5.250 - 0.375$$

$$= 512 - 89.750 - 0.375$$

$$= 512 - 90.125$$

$$= 421.875$$

$\therefore P(7.5) = 421.875$

Fig (1) may be written as.

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	1				
2	8	7			
3	27	19	12		
4	64	37	18	6	
5	125	61	24	6	0
6	216	91	30	6	0
7	343	127	36	6	0
8	512	169	42	6	0

Algorithm to implement newtons backward ~~interpolation formula~~ difference table.

1. take all $y=f(x)$ values in an array 'a[][]' where n is the total no. of elements.
2. $i=2$; $i>2$,
3. Repeat step 4 to 9 untill $j<=n$, which starts from 'i', $j=3$, incremented by 1.
4. Repeat step 5 to 6 untill ~~K...~~ $K=i$, starts with $K=n$, and decremented by '1'
5. set $a[K][j] = a[K][j-1] - a[K-1][j-1]$.
6. ~~print a[K][j], and goto next line.~~
7. end of step 4 loop.
8. ~~print "double space"~~ for, $K=i-1$ to 1, set $a[K][j]=0$;
8. set $i=i+1$;
9. end of step 3 loop
10. print $a[K,j]$.
11. exit.

Algorithm for obtaining estimated value using ~~for...~~ backward interpolation formula.

1. Get the all x_i values.
2. Obtained the equal interval width $h=x_{i+1}-x_i$
3. let x_n be the last value among all x_i's
4. Take the value 'p' as input for which we want to estimate $f(p)$.
5. set $u=\frac{p-x_n}{h}$
6. set $sum = y_n = f(x_n)$.
7. use the above algorithm and obtained the backward difference table.

~~(8) Set ...~~

(8) Repeat step 10 to 12 untill $j <= n-1$, j is incremented, j starts with, $j=3$

(9) Set $t=1$
Repeat step 10, untill ucount $<= j-1$, ucount increments by 1, ucount starts with '1'.

(10) Set $t = t*u$; set $u = u+1$.

(10a) end of step (9) loop.

(11) Set $p = t * \frac{a[n][j]}{factorial(j-1)}$

~~(12) Set u = u+1~~ sum = sum + p,

(13) end of step 9 loop.

(14) print the value of sum.

(15) exit.

Q. Given following data in table format.

x	;	0	5	10	15	20
$f(x)$	:	1.0	1.6	3.8	8.2	15.4

∴ From these data obtain backward difference table and compute $f(21)$.

x	$y=f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	1.0				
5	1.6	0.6			
10	3.8	2.2	1.6		
15	8.2	4.4	2.2	0.6	0
20	15.4	7.2	2.8	0.6	

now, Here $x_n = 20$, $h = (x_n - x_{n-1}) = (20-15) = 5$

∴ $h = 5$.

and $x = 21$, given.

∴ $u = \frac{x - x_n}{h} = \frac{21-20}{5} = \frac{1}{5} = 0.2$.

now from newtons backward interpolation formula, we have,