Calculus of finite difference

Suppose you are given a table of values and you have to determine a value of y, ~~into~~ from the given 'x', but $f(x)$ is not mentioned yet. Then for getting the value of ~~x~~ y (called estimated or representative value in connection with given x) ~~the~~ we use the finite difference calculus which deals with the variations in the values of the function due to change of the independent variable (x).

Forward difference:-

Let us consider a function $y = f(x)$.

value of ~~x~~ x ~~is~~ within $a \le x \le b$.

here $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ..., $x_r = x_0 + rh$, ..., $x_{n-1} = x_0 + (n-1)h$, $x_n = x_0 + nh = b$.

now, $y_0 = f(x_0)$ → arguments.

$$y_1 = f(x_1) = f(x_0 + h)$$
$$y_2 = f(x_2) = f(x_0 + 2h)$$
$$\vdots$$
$$y_r = f(x_r) = f(x_0 + rh)$$
$$\vdots$$
$$y_n = f(x_n) = f(x_0 + nh)$$

Now, the first order difference is defined as

$$\Delta f(x_0) = f(x_0 + h) - f(x_0) = y_1 - y_0 = \Delta y_0$$
$$\Delta f(x_0 + h) = f(x_0 + 2h) - f(x_0 + h) = y_2 - y_1 = \Delta y_1$$
$$\vdots$$
$$\Delta f(x_0 + (n-1)h) = f(x_0 + nh) - f(x_0 + (n-1)h) = y_n - y_{n-1} = \Delta y_{n-1}$$

The second order difference are denoted by.

$$\Delta^2 f(x_0) = \Delta^2 y_0 = \Delta f(x_0+h) - \Delta f(x_0)$$
$$= f(x_0+2h) - f(x_0+h) - f(x_0+h) + f(x_0)$$
$$= f(x_0+2h) - 2f(x_0+h) + f(x_0)$$
$$= y_2 - 2y_1 + y_0 = \Delta^2 y_0.$$

$$\Delta^2 f(x_0+h) = \Delta^2 y_1 = \Delta f(x_0+2h) - \Delta f(x_0+h)$$
$$= f(x_0+3h) - f(x_0+2h) - f(x_0+2h) + f(x_0+h)$$
$$= f(x_0+3h) - 2f(x_0+2h) + f(x_0+h)$$
$$= y_3 - 2y_2 + y_1 = \Delta^2 y_1$$

Similarly $\Delta^2 f(x_0+2h) = y_4 - 2y_3 + y_2 = \Delta^2 y_2$

and $\Delta^3 f(x_0) = \Delta^2 f(x_0+h) - \Delta^2 f(x_0)$

$$= \Delta f(x_0+2h) - \Delta f(x_0+h) - \Delta f(x_0+h) + \Delta f(x_0)$$
$$= f(x_0+3h) - f(x_0+2h) - f(x_0+2h) + f(x_0+h) - f(x_0+2h) + f(x_0+h) + f(x_0+h) - f(x_0)$$
$$= f(x_0+3h) - 3f(x_0+2h) + 3f(x_0+h) - f(x_0)$$
$$= y_3 - 3y_2 + 3y_1 - y_0 = \Delta^3 y_0$$

In general $\therefore$ $\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x).$

Q Suppose we are given a set of values,

x :	0	1	2	3	4	5
$f(x)$:	12	15	20	27	39	52

then the forward difference table is given by. ——

x	$y=f(x)$	Δy_r	$\Delta^2 y_r$	$\Delta^3 y_r$	$\Delta^4 y_r$	$\Delta^5 y_r$
						-10.
0	12	3	2	0	3	
1	15	5	2	3	-7	
2	20	7	5	-4		
3	27	12	1			
4	39	13				
5	52					

$\Delta^3 y_1 = 3$, $\nabla^3 y_4 = 3$

$\nabla^2 y_1 = 2$, $\Delta^2 y_0 = 2$

$\Delta^3 y_1 = \nabla^3 y_4$;

$\Delta^2 y_0 = \nabla^2 y_2$; $\Delta^i_j = \nabla^i_{i+j}$, $\Delta^i_j = \nabla^i_{i+j}$

$\therefore \Delta^2 f(0) = 2, \quad \Delta^3 f(1) = 3$ etc.

Backward difference

Backward difference operator is denoted by '∇'.

if $y=f(x)$ be a function and value of x is with a and b,

First order backward difference

$$\nabla f(x_0) = f(x_0) - f(x_0 - h)$$

$$\nabla f(x_0+h) = f(x_0+h) - f(x_0) = y_1 - y_0 = \nabla y_1$$

$$\nabla f(x_0+2h) = f(x_0+2h) - f(x_0+h) = y_2 - y_1 = \nabla y_2$$

$$\cdots\cdots\cdots$$

$$\nabla f(x_0+nh) = f(x_0+nh) - f(x_0+(n-1)h) = y_n - y_{n-1} = \nabla y_n.$$

2nd order backward difference.

$$\nabla^2 f(x_0+2h) = \nabla f(x_0+2h) - \nabla f(x_0+h)$$

$$= f(x_0+2h) - f(x_0+h) - f(x_0+h) + f(x_0)$$

$$= f(x_0+2h) - 2f(x_0+h) + f(x_0)$$

$$= y_2 - 2y_1 + y_0 = \nabla^2 y_2$$

$$\nabla^2 f(x_0+3h) = \nabla f(x_0+3h) - \nabla f(x_0+2h)$$

$$= f(x_0+3h) - f(x_0+2h) - f(x_0+2h) + f(x_0+h)$$

$$= f(x_0+3h) - 2f(x_0+2h) + f(x_0+h)$$

$$= y_3 - 2y_2 + y_1 = \nabla^2 y_3$$

$$\nabla^3 f(x_0+3h) = \nabla^2 f(x_0+3h) - \nabla^2 f(x_0+2h)$$

$$= \{\nabla f(x_0+3h) - \nabla f(x_0+2h)\} - \{\nabla f(x_0+2h) - \nabla f(x_0+h)\}$$

$$= \{f(x_0+3h) - f(x_0+2h) - f(x_0+2h) + f(x_0+h)\} - \{f(x_0+2h) - f(x_0+h) - f(x_0+h) + f(x_0)\}$$

$$= \{f(x_0+3h) - 2f(x_0+2h) + f(x_0+h)\} - \{f(x_0+2h) - 2f(x_0+h) + f(x_0)\}$$

$$= f(x_0+3h) - 2f(x_0+2h) + f(x_0+h) - f(x_0+2h) + 2f(x_0+h) - f(x_0)$$

$$= f(x_0+3h) - 3f(x_0+2h) + 3f(x_0+h) - f(x_0)$$

$$= y_3 - 3y_2 + 3y_1 - y_0 = \nabla^3 y_3$$

In general.

$$\nabla^n f(x) = \nabla^{n-1} f(x) - \nabla^{n-1} f(x-h).$$

Q. Write down backward difference table for same table given in last page.

x	$y=f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
0	12					
1	15	3				
2	20	5	2			
3	27	7	2	0		
4	39	12	5	3	3	
5	52	13	1	-4	-7	-10

$\nabla^2 f(5) = 1$, $\nabla^4 f(4) = 3$ etc.

from these two table we conclude that $\Delta^i y_j = \nabla^i y_{i+j}$

properties of forward difference operator Δ.

1. difference of a constant is zero

if $f(x) = c$ $\therefore$ $f(x+h) = c$

$\therefore$ $\Delta f(x) = f(x+h) - f(x)$

$= c - c$

$= 0$.

$\therefore$ $\Delta f(x) = 0$

$\therefore$ $\Delta c = 0$.

We can see from forward & backward difference table that.

$\nabla^2 y_6 = \Delta^2 y_4$, $\nabla^3 y_4 = \Delta^3 y_1$

$\nabla^4 y_5 = \Delta^4 y_1$, $\nabla y_3 = \Delta y_2$

$\nabla^i y_j = \Delta^i y_{j-i}$

i.e. $\nabla^i y_{i+j} = \Delta^i y_j$

2. Commutative with respect to constant, i.e.

$$\Delta\left[K f(x)\right] = K . \Delta f(x).$$

Let $\phi(x) = K f(x)$. $\therefore$ $\phi(x+h) = K . f(x+h)$

$\therefore$ $\Delta \phi(x) = \phi(x+h) - \phi(x)$

$= K f(x+h) - K f(x)$

$= K\left[f(x+h) - f(x)\right]$

$= K . \Delta f(x)$

2. Distributive :-

if $f(x)$, $\phi(x)$, $\psi(x)$, --- are functions of x,

then $\Delta\left[f(x) \pm \phi(x) \pm \psi(x) \pm \text{---}\right] = \Delta f(x) \pm \Delta\phi(x) \pm \Delta\psi(x)$ $\pm$ ---

Let $F(x) = f(x) \pm \phi(x) \pm \psi(x) \pm$ ---

$\therefore$ $\Delta F(x) = F(x+h) - F(x)$

$= \{f(x+h) \pm \phi(x+h) + \psi(x+h) \pm \text{----}\}$

$- \{f(x) \pm \phi(x) \pm \psi(x) \pm \text{----}\}$

$= \{f(x+h) - f(x)\} \pm \{\phi(x+h) - \phi(x)\} \pm \{\psi(x+h) - \psi(x)$ ---

$\Delta f(x) \pm \Delta \phi(x) \pm \Delta \psi(x) \pm \cdots \cdots$

Thus $\Delta\left[K f(x) \pm \lambda \phi(x) \pm \mu \psi(x) \pm \cdots\right]$

$$= K \Delta f(x) \pm \lambda \Delta \phi(x) \pm \mu \Delta \psi(x) \pm \cdots \cdots$$

4 $\Delta^m . \Delta^n f(x) = \Delta^{m+n} f(x)$

5 $\Delta\left[f(x) . \phi(x)\right]$

$$= f(x+h) . \phi(x+h) - f(x) . \phi(x)$$

$$\cancel{= f(x+h) . \phi(x+h) - f(x+h) . \phi(x) + f(x+h) . \phi(x) - f(x) . \phi(x)}$$

$$\cancel{= f(x+h) \{\phi(x+h) - \phi(x)\}}$$

$$= f(x+h) . \phi(x+h) - \phi(x+h) . f(x) + \phi(x+h) . f(x) - f(x) . \phi(x)$$

$$= \phi(x+h) . \left(f(x+h) - f(x)\right) + f(x)\left(\phi(x+h) - \phi(x)\right)$$

$$= \phi(x+h) . \Delta f(x) + f(x) . \Delta \phi(x)$$

$$= \Delta f(x) . \phi(x+h) + \Delta \phi(x) . f(x) = \Delta\left[f(x) . \phi(x)\right] \quad \text{---} (1)$$

Again, $\Delta\left[f(x) . \phi(x)\right]$

$$= f(x+h) . \phi(x+h) - f(x) . \phi(x)$$

$$= f(x+h) . \phi(x+h) - f(x+h) . \phi(x) + f(x+h) . \phi(x) - f(x) \phi(x)$$

$$= f(x+h) . \{\phi(x+h) - \phi(x)\} + \phi(x)\{f(x+h) - f(x)\}$$

$$= \Delta \phi(x) . f(x+h) + \phi(x) . \Delta f(x)$$

$$\therefore \Delta\left[f(x) . \phi(x)\right] = \Delta f(x) . \phi(x) + \Delta \phi(x) . f(x+h) \quad \text{---} (2)$$

6. $\Delta\left[\frac{f(x)}{\phi(x)}\right] = \left\{ \frac{f(x+h)}{\phi(x+h)} - \frac{f(x)}{\phi(x)} \right.$

$$= \frac{f(x+h)\cdot\phi(x) - f(x)\cdot\phi(x+h)}{\phi(x+h)\cdot\phi(x)}$$

$$= \frac{f(x+h)\cdot\phi(x) - f(x)\cdot\phi(x) + f(x)\cdot\phi(x) - f(x)\cdot\phi(x+h)}{\phi(x+h)\cdot\phi(x)}$$

$$= \frac{\phi(x)\{f(x+h)-f(x)\} - f(x)\{\phi(x+h) - \phi(x)\}}{\phi(x+h)\cdot\phi(x)}$$

$$= \frac{\Delta f(x)\cdot\phi(x) - \Delta\phi(x)\cdot f(x)}{\phi(x+h)\cdot\phi(x)}. \quad [\phi(x) \neq 0]$$

Q. prove that,

$$\Delta\cdot\nabla = \Delta - \nabla$$

now. $\Delta\cdot\nabla f(x)$

$= \Delta\{\nabla f(x)\}$

$= \Delta\{f(x) - f(x-h)\}$

$= \Delta f(x) - \Delta f(x-h)$.

$= \Delta f(x) - \{f(x) - f(x-h)\}$

$= \Delta f(x) - \{\nabla f(x)\}$

$= \Delta f(x) - \nabla f(x)$

$= (\Delta - \nabla) f(x)$.

Fundamental theorem of difference calculus.

If $f(x)$ be a polynomial, of degree n, then n order difference is constant and $n+1$ th order difference is vanished.

Shift operator E, and ~~relation~~ difference operator Δ, relation:-

for ~~as~~ a function $f(x)$, the shift operator 'E' is defined as

$$Ef(x) = f(x+h)$$

$$\therefore Ef(x) - f(x) = f(x+h) - f(x)$$

$$\therefore (E-1) f(x) = \Delta f(x)$$

$$\therefore E - 1 = \Delta$$

$$\therefore E = 1 + \Delta.$$

now. $E^2 f(x) = f(x+2h)$

$E^3 f(x) = f(x+3h)$

— — — —

— — — —

$E^n f(x) = f(x+nh)$

$E^{-n} f(x) = f(x-nh)$

shift operator obeys the following rules.

(a) if $u(x) = f(x) + \phi(x) + \psi(x) + \cdots$

$Eu(x) = u(x+h)$

$= f(x+h) + \phi(x+h) + \psi(x+h) + \cdots$

$= Ef(x) + E\phi(x) + E\psi(x) + \cdots$

(b) if $u(x) = Kf(x)$

then $Eu(x) = \cancel{E}\, K f(x) = u(x+h)$

$= \cancel{E}\, K f(x+h) = K..f(x+h)$

$= K.E f(x+h) = K.E f(x).$

$E\{Kf(x)\} = KEf(x).$

Q prove that

$$E.\Delta = \Delta.E$$

now $E\Delta f(x)$

$$= E\{\Delta f(x)\}$$

$$= E\{f(x+h) - f(x)\}$$

$$= \{Ef(x+h) - Ef(x)\}$$

$$= Ef(x+h) - f(x+h)$$

$$= (E-1) f(x+h)$$

$$= \Delta f(x+h)$$

$$= \Delta.E f(x)$$

$$\therefore E.\Delta = \Delta.E$$

Relation between difference operator Δ and $D \left(= \frac{d}{dx}\right)$ of differential calculus.

By Taylor's theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

$$= f(x) + h.Df(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \cdots$$

$$= \left(1 + hD + \frac{h^2}{2!} D^2 + \frac{h^3 . D^3}{3!} + \cdots\right) f(x)$$

$$\therefore Ef(x) = e^{hD} f(x)$$

$$\therefore E = e^{hD}$$

$$\therefore 1 + \Delta = e^{hD}$$

$$\therefore hD = \log(1+\Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots$$

$$\therefore D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right]$$

now

$f(x_0+nh)$

$= E^n f(x_0) = (1+\Delta)^n f(x_0)$

$= [1 + {}^nC_1 \Delta + {}^nC_2 \Delta^2 + {}^nC_3 \Delta^3 + \cdots + {}^nC_r \Delta^r + \cdots + {}^nC_n \Delta^n] f(x_0)$

$= f(x_0) + {}^nC_1 \Delta f(x_0) + {}^nC_2 \Delta^2 f(x_0) + {}^nC_3 \Delta^3 f(x_0) + \cdots + {}^nC_n \Delta^n f(x_0)$

$= y_0 + {}^nC_1 \Delta y_0 + {}^nC_2 \Delta^2 y_0 + {}^nC_3 \Delta^3 y_0 + \cdots + {}^nC_n \Delta^n y_0.$

$= y_0 + n.\Delta y_0 + \frac{n(n-1)}{2!}.\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \cdots$

Q. Find the polynomial $f(x)$, which satisfy the following data and hence find the value of $f(1.5)$.

x :	1	2	3	4	5
$y = f(x)$:	4	13	34	73	136

Solution. We have to find $f(1.5)$ i.e, the value of $f(x)$ corresponding to $x = 1.5$,

As $x = 1.5$ ~~is in the begining side~~ resides in the front portion of the interval $1 \le x \le 5$.

Then we have to form, the forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	4	9	12	6	0
2	13	21	18	6	
3	34	39	24		
4	73	63			
5	136				

Let, if we shift the value $x_0 = 1$, to its nth position

then it will reach $x_0 + nh = x = 1.5$

now $x_0 = 1$, $h = 1$.

$\therefore 1 + n.1 = x$

$\therefore n = (x-1)$.

now, we know that

$f(x) = f(x_0 + nh) = E^n f(x_0)$

$= (1+\Delta)^n f(x_0)$

$= (1+\Delta)^{(x-1)} f(x_0)$

$= f(x_0) + (x-1)\Delta f(x_0) + \frac{(x-1)(x-2)}{2!}\Delta^2 f(x_0) + \frac{(x-1)(x-2)(x-3)}{3!}\Delta^3 f(x_0)$

[rest of the te = 0]

$= f(x_0) + (x-1)\Delta y_0 + \frac{(x-1)(x-2)}{2}.\Delta^2 y_0 + \frac{(x-1)(x-2)(x-3)}{6}.\Delta^3 y_0$

$= f(x_0) + (x-1).9 + \frac{(x-1)(x-2)}{\cancel{2}}.\cancel{12}^{6} + \frac{(x-1)(x-2)(x-3)}{\cancel{6}}.\cancel{6}$

$= f(x_0) + (x-1).9 + 6(x-1)(x-2) + (x-1)(x-2)(x-3)$

$= f(x_0) + (x-1).9 + (x-1)(x-2)\{6 + x - 3\}$

$= f(x_0) + (x-1).9 + (x-1)(x-2)(x+3)$

$= f(x_0) + (x-1)\{9 + (x-2)(x+3)\}$

$= f(x_0) + (x-1)\{9 + x^2 + x - 6\}$

$= f(x_0) + (x-1)\{x^2 + x + 3\}$

$\therefore f(x) = f(x_0) + (x-1)\{x^2 + x + 3\} = x^3 + 2x + 1$

$\therefore f(1.5) = 4 + (1.5-1)\{(1.5)^2 + 1.5 + 3\}$

$= 4 + \frac{1}{2} \cdot \{2.25 + 4.5\}$

$= 4 + \frac{1}{2} \cdot 6.750$

$= 4 + 3.375$ ~~4.375~~

$= 7.375$. Ans.

Q. Write down the polynomial of degree three relevant to the data;

x	:	-1	0	1	2
$y = f(x)$	:	1	1	1	-5

Ans:- The forward difference table is given below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	1	0	0	-6
0	1	0	-6	
1	1	-6		
2	-5			

now. $x_0 = -1$, $f(x_0) = y_0 = 1$, $h = 1$.

$x_0 + nh = x$

$\therefore -1 + n \cdot 1 = x \qquad \therefore x = n - 1$

$\therefore n = (x+1)$.

now $f(x) = f(x_0 + nh) = E^n f(x_0) = (1+\Delta)^n f(x_0)$

$= \left[1 + n\Delta + \frac{n(n-1)}{2}\Delta^2 + \frac{n(n-1)(n-2)}{6}\Delta^3\right] f(x_0)$

$= f(x_0) + x\,\Delta f(x_0) + \frac{x(x-1)}{2}\Delta^2 f(x_0) + \frac{x(x-1)(x-2)}{6}\Delta^3 f(x_0)$

$= f(x_0) + x\,\Delta y_0 - \frac{x(x-1)}{2}\Delta^2 y_0 - \frac{x(x-1)(x-2)}{6}\Delta^3 y_0$

$= y_0 + (x+1)\cdot 0 - \frac{x(x-1)}{2}\cdot 0 + \frac{x(x-1)(x-2)}{6}\Delta^3 y_0$

$= y_0 + \frac{(x+1)\cdot x\,(x-1)}{6}\cdot(-6)$

$= 1 + \frac{(x+1)\cdot x\,(1-x)}{6}\cdot 6$

$= 1 + x(1-x^2)$

$= 1 + x - x^3 = f(x)$

$\therefore\ f(x) = 1 + x - x^3$ Ans

Q. Evaluate the missing term in the following table

x :	0	1	2	3	4	5
$y = f(x)$:	0	–	8	15	–	35

Ans. As the four values are present corresponding to each x, then we take our polynomial is of degree $(n-1)$, i.e. $=(4-1)=3$. then 4th order difference will be vanished

$\Delta^4 f(x_0) = \Delta^4 y_0$

now $\Delta^4 f(x_0) = 0$

$\therefore (E-1)^4 f(x_0) = 0$

$\therefore [E^4 - 4E^3 + 6E^2 - 4E + 1]\, f(x_0) = 0$

$\therefore E^4 f(x_0) - 4E^3 f(x_0) + 6E^2 f(x_0) - 4Ef(x_0) + f(x_0) = 0$

or, $f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x_0+1) + f(x_0) = 0$ ---- (1)

Now by putting $x = 0$ in equation (i) we get,

$$f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$$

or, $f(4) - 4.15 + 6.8 - 4f(1) + 0 = 0$

or $f(4) - 4f(1) = 60 - 48 = 12$

$\therefore$ $f(4) - 4f(1) = 12$ ---- (ii)

and by putting $x = 1$ in equation (i) we get

$$f(5) - 4f(4) + 6f(3) - 4f(2) + f(1) = 0$$

or, $35 - 4.f(4) + 6.15 - 4.8 + f(1) = 0$

or $f(1) - 4f(4) = -93$

~~or $4f(4) - f(1) = 93$ ---- (iii)~~

$4f(4) - f(1) = 93$

$f(4) - 4f(1) = 12$ ---- $\times 4$

$4f(4) - 4f(1) = 372$

or $4f(4) - 16f(1) = 48$

$-12f(1) = 324$

6492

$f(1) = -32.$

$\therefore$ $4f(4) - f(1) = 93$ ---- (iii)

$f(4) - 4f(1) = 12$ ---- (ii) ×

$$4f(4) - f(1) = 93$$
$$4f(4) - 16f(1) = 48$$
$$- \quad + \quad -$$

$$15f(1) = 45$$
$$\therefore f(1) = 3.$$

And, $f(4) - 4f(1) = 12$

$$\Rightarrow f(4) - 4.3 = 12$$

$$\Rightarrow f(4) = 12 + 12 = 24$$

$\therefore$ So the Answer $\div$ $f(4) = 24$, $f(1) = 3$.

Q. Prove that,

$$\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

Answer $\Delta \log f(x)$

$$= \log f(x+h) - \log f(x)$$

$$= \log \frac{f(x+h)}{f(x)}$$

$$= \log \frac{f(x+h) - f(x) + f(x)}{f(x)}$$

$$= \log \frac{\Delta f(x) + f(x)}{f(x)}$$

$$= \log\left[\frac{\Delta f(x)}{f(x)} + 1\right]$$

$$= \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

Q. Evaluate $\left(\frac{\Delta^2}{E}\right) x^3$ where $h = 1$

Ans $\left(\frac{\Delta^2}{E}\right)x^3$

$= \frac{(E-1)^2}{E}x^3$

$= \left(\frac{E^2-2E+1}{E}\right)x^3$

$= \left(E-2+\frac{1}{E}\right)x^3$

$= E(x^3) - 2x^3 + \frac{1}{E}(x^3)$

$= (x+1)^3 - 2x^3 + (x-1)^3$

$= \cancel{x^3} + \cancel{3x^2} + 3x \cancel{+1} \cancel{-2x^3} + \cancel{x^3} - \cancel{3x^2} + 3x - \cancel{1}$

$= 6x.$

Q. Show that $\Delta^n[Ke^{ax}] = K(e^{ah}-1)^n e^{ax}$

Ans $\Delta[Ke^{ax}] = Ke^{a(x+h)} - K.e^{ax}$

$= Ke^{ax}(e^{ah}-1)$

$\Delta^2[Ke^{ax}] = \Delta Ke^{a(x+h)} - \Delta K.e^{ax}$

$= Ke^{a(x+h)}(e^{ah}-1) - Ke^{ax}(e^{ah}-1)$

$= K(e^{ah}-1).e^{ax}.(e^{ah}-1)$

$= Ke^{ax}(e^{ah}-1)^2.$

$\therefore \Delta^n[Ke^{ax}] = Ke^{ax}(e^{ah}-1)^n.$

Problems: