Numerical ~~Analy~~ Differentiation

It is the process of finding approximate numerical value of the derivative or derivatives of a function $f(x)$, for a particular given value of 'x', when the $f(x)$ is not defined explicitly, we fit a suitable polynomial to the given set of points and makes derivative of it as many times we ~~desire~~ ~~desire~~ need. So in two case we ~~use~~ use this numerical differentiation

1) when functions are in set of points or tabulated form

2) when determining derivative is very complicated in analytical form.

So the method of obtaining derivatives in numerical process is called numerical differentiation. Here we use newtons forward and backward formula only.

Consider Newton's forward difference formula.

$$y = y_0 + u.\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

by differentiating both side with respect to 'u'.

we have.

$$\frac{dy}{du} = 0 + \Delta y_0 . 1 + \frac{\Delta^2 y_0}{2!}(2u-1) + \frac{3u^2-6u+2}{6}.\Delta^3 y_0 + \cdots$$

$$\therefore \frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}.\Delta^2 y_0 + \frac{3u^2-6u+2}{6}.\Delta^3 y_0 + \cdots$$

Again ~~$x = x_0 + uh$~~ $\frac{x - x_0}{h} = u$

$\therefore x - x_0 = uh.$

$\therefore x = x_0 + uh.$

$\therefore$ by differentiating both side with respect to x, we have

$$1 = 0 + \frac{du}{dx}.h$$

$$\text{or, } 1 = h.\frac{du}{dx} \quad // \quad \frac{du}{dx} = \frac{1}{h}$$

now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{du}{dx} \cdot \frac{dy}{du}$

now, we can put the value of $\frac{du}{dx}$ and $\frac{dy}{du}$, and we have

$$\frac{dy}{dx} = \frac{1}{h} \cdot \left[\Delta y_0 + \frac{2u-1}{2} \Delta^2 y_0 + \frac{3u^2-6u+2}{6} \Delta^3 y_0 + \cdots \right]$$

therefore

$$f'(x) = \frac{1}{h} \left[\Delta y_0 + \frac{2u-1}{2} \cdot \Delta^2 y_0 + \frac{3u^2-6u+2}{6} \cdot \Delta^3 y_0 + \cdots \right]$$

This formula is used to compute numerical values of derivatives of $f(x)$.

if $x = x_0$, then $u = \frac{x-x_0}{h} = \frac{x_0 - x_0}{h} = \frac{0}{h} = 0$.

$u = 0$

then $$f'(x_0) = \frac{1}{h} \left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots \right]$$

Again, $$f'(x) = \frac{1}{h} \left[\Delta y_0 + \frac{2u-1}{2} \Delta^2 y_0 + \frac{3u^2-6u+2}{6} \Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24} \cdot \Delta^4 y_0 + \cdots \right]$$

now by differentiating with respect to u we have

$$f''(x) \frac{dx}{du} = \frac{1}{h} \left[0 + (1-0) \cdot \Delta^2 y_0 + \frac{6u-6}{6} \Delta^3 y_0 + \frac{12u^2-36u+22}{24} \Delta^4 y_0 + \cdots \right]$$

$$\therefore f''(x) \cdot \frac{dx}{du} = \frac{1}{h} \left[\Delta^2 y_0 + (u-1) \Delta^3 y_0 + \frac{6u^2-18u+11}{12} \cdot \Delta^4 y_0 + \cdots \right]$$

or, $$f''(x) \cdot \frac{1}{\frac{du}{dx}} = \frac{1}{h} \left[\Delta^2 y_0 + (u-1) \Delta^3 y_0 + \frac{6u^2-18u+11}{12} \Delta^4 y_0 + \cdots \right]$$

$$\therefore f''(x) = \frac{1}{h} \cdot \frac{du}{dx} \left[\Delta^2 y_0 + (u-1) \Delta^3 y_0 + \frac{6u^2-18u+11}{12} \Delta^4 y_0 + \cdots \right]$$

$\therefore f''(x) = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \cdots\right]$

if $x = x_0$ then $u = 0$.

$\therefore f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$

Again.

$$f''(x) = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \cdots\right]$$

now differentiating both side with respect to 'u' we have.

$$f'''(x)\frac{dx}{du} = \frac{1}{h^2}\left[0 + \Delta^3 y_0 + \frac{12u-18}{12}\Delta^4 y_0 + \cdots\right]$$

$$\therefore f'''(x) = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{2u-3}{2}\Delta^4 y_0 + \cdots\right]$$

now, if $x = x_0$ then $u = 0$.

$$\therefore f'''(x_0) = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \frac{7}{4}\Delta^5 y_0 - \cdots\right]$$

like this way we can obtained upto $f^{n}(x_0)$

Estimation of Error in Differentiation formula based on Newton's forward difference formula.

We know that error committed in newtons forward interpolation formula is.

$$R_{n+1}(x) = u(u-1)(u-2)\cdots(u-n).h^{n+1}.\frac{f^{n+1}(\xi)}{(n+1)!}$$

where $x_0 \le x \le x_n$.

therefore differentiating both side with respect to u we have

$$R'_{n+1}(x) = \frac{h^{n+1}}{(n+1)!}\left[\left\{\frac{d}{du}u(u-1)(u-2)\cdots(u-n)\right\}.f^{n+1}(\xi).\frac{du}{dx} + u(u-1)(u-2)\cdots(u-n).f^{n+2}(\xi)\right]$$

now. $\frac{x-x_0}{h} = u$.

$\therefore x - x_0 = u.h$

$\therefore \frac{dx}{du} = h$ [by differentiating with respect to u]

$\therefore \frac{du}{dx} = \frac{1}{h}$

$$\therefore R'_{n+1}(x) = \frac{h^{n+1}}{(n+1)!} \cdot \frac{1}{h}\left\{\frac{d}{du} u(u-1)(u-2)\cdots(u-n)\right\} f^{n+1}(\xi)$$

$$+ h^{n+1} u(u-1)(u-2)\cdots(u-n) \cdot \frac{f^{n+2}(\xi)}{(n+1)!}$$

Now we also know that,

$$f[x_0, x_1, x_2, x_3, \cdots x_n]$$

$$= \frac{\Delta^n y_0}{n!\, h^n} = \frac{\Delta^n f(x_0)}{n!\, h^n}$$

and $f[x_0, x_1, x_2 \cdots x_n] = \frac{1}{n!} f^{n}(x_0) = \frac{f^n(x_0)}{n!}$

$$\therefore \frac{\Delta^n f(x_0)}{n!\, h^n} = \frac{f^n(x_0)}{n!}$$

$$\therefore \frac{\Delta^n f(x)}{\cancel{n!}\, h^n} = \frac{f^n(x)}{\cancel{n!}} \qquad \therefore \Delta^n f(x) = h^n f^n(x)$$

$$\therefore R'_{n+1}(x) = \frac{h^n}{(n+1)!} \cdot \frac{d}{du}\{u(u-1)(u-2)\cdots(u-n)\} \cdot f^{n+1}(\xi)$$

$$+ h^{n+1}\{u(u-1)(u-2)\cdots(u-n)\} \cdot \frac{f^{n+2}(\xi)}{(n+1)!}$$

$$= \frac{1}{h} \cdot \frac{d}{du}\{u(u-1)(u-2)\cdots(u-n)\} \cdot \frac{h^{n+1} f^{n+1}(\xi)}{(n+1)!}$$

$$+ \frac{1}{h} \cdot \{u(u-1)(u-2)\cdots(u-n)\} \cdot \frac{h^{n+2} f^{n+2}(\xi)}{(n+1)!}$$

$$= \frac{1}{h}\left[\frac{d}{du}\{u(u-1)(u-2)\cdots(u-n)\} \cdot \frac{\Delta^{n+1} f(x_0)}{(n+1)!} + \{u(u-1)(u-2)\cdots(u-n)\} \cdot (n+2) \cdot \frac{\Delta^{n+2} f(x_0)}{(n+2)!}\right]$$

Now,

$$u(u-1)(u-2)(u-3)\cdots(u-n)$$

$$= \frac{u(u-1)(u-2)(u-3)\cdots(u-n)(u-n-1)(u-n-2)\cdots 3.2.1}{(u-n-1)(u-n-2)(u-n-3)\cdots 3.2.1}$$

$$= \frac{u!}{\{u-(n+1)\}!}$$

$$\therefore \frac{1}{h}\left[\frac{d}{du}\left\{u(u-1)(u-2)\cdots(u-n)\right\}\frac{\Delta^{n+1}f(x_0)}{(n+1)!}\right.$$

$$\left. + \left\{u(u-1)(u-2)\cdots(u-n)\right\}\frac{\Delta^{n+2}f(x_0)}{(n+1)!}\right]$$

$$= \frac{1}{h}\left[\frac{d}{du}\left\{\frac{u!}{\{u-(n+1)\}!}\cdot\frac{1}{(n+1)!}\right\}\Delta^{n+1}f(x_0)\right.$$

$$\left. + \left\{\frac{u!}{\{u-(n+1)\}!}\cdot\frac{1}{(n+1)!}\right\}\Delta^{n+2}f(x_0)\right]$$

$$= \frac{1}{h}\left[\frac{d}{du}\left\{{}^{u}C_{n+1}\right\}\Delta^{n+1}f(x_0) + \left\{{}^{u}C_{n+1}\right\}\Delta^{n+2}f(x_0)\right]$$

$$= \frac{d}{du}\left\{{}^{u}C_{n+1}\right\}\frac{\Delta^{n+1}f(x_0)}{h} + \left\{{}^{u}C_{n+1}\right\}\frac{\Delta^{n+2}f(x_0)}{h}$$

at $x = x_0$, we know $u = 0$, because $u = \frac{x - x_0}{h}$,

2nd term is totally vanished

So, $$R'_{n+1}(x_0) = \frac{d}{du}\left\{{}^{u}C_{n+1}\right\}_{u=0}\frac{\Delta^{n+1}f(x_0)}{h} + \left\{{}^{u}C_{n+1}\right\}_{u=0}\frac{\Delta^{n+2}f(x_0)}{h}$$

$$= \frac{d}{du}\left\{{}^{u}C_{n+1}\right\}_{u=0}\frac{\Delta^{n+1}f(x_0)}{h} + \left[\frac{u(u-1)(u-2)\cdots(u-n)}{(n+1)!}\cdot\frac{\Delta^{n+2}f(x_0)}{h}\right]_{u=0}$$

$$= \frac{d}{du}\left\{{}^{u}C_{n+1}\right\}_{u=0}\frac{\Delta^{n+1}f(x_0)}{h} + 0$$

$$= \frac{d}{du}\left\{{}^{u}C_{n+1}\right\}_{u=0}\frac{\Delta^{n+1}f(x_0)}{h}$$

$$= \frac{d}{du}\left\{{}^{u}C_{n+1}\right\}_{u=0}\frac{\Delta^{n+1}f(x_0)}{h}$$

$$= \frac{d}{du}\left[\frac{u(u-1)(u-2)\cdots(u-n)}{(n+1)!}\right]_{u=0}\cdot\frac{\Delta^{n+1}f(x_0)}{h}$$

$$= \frac{d}{du}\left[u(u-1)(u-2)\cdots(u-n)\right]_{u=0}\frac{\Delta^{n+1}f(x_0)}{h\cdot(n+1)!}$$

$$= \left[u\cdot\frac{d}{du}\{(u-1)(u-2)\cdots(u-n)\} + (u-1)(u-2)\cdots(u-n)\cdot\frac{d}{du}u\right]_{u=0}\cdot\frac{\Delta^{n+1}f(x_0)}{h\cdot(n+1)!}$$

$$= \left[0\cdot\frac{d}{du}\{(u-1)(u-2)\cdots(u-n)\} + \{(-1)(-2)(-3)\cdots(-n)\}\right]\cdot\frac{\Delta^{n+1}f(x_0)}{h\cdot(n+1)!}$$

$$= \left[0 + (-1)^n\cdot n!\right]\cdot\frac{\Delta^{n+1}f(x_0)}{h\cdot(n+1)!} = (-1)^n\cdot n!\cdot\frac{\Delta^{n+1}f(x_0)}{h\cdot(n+1)!}$$

$$\therefore R'_{n+1}(x_0) = (-1)^n \cdot \frac{\Delta^{n+1} f(x_0)}{(n+1)\cdot h}$$

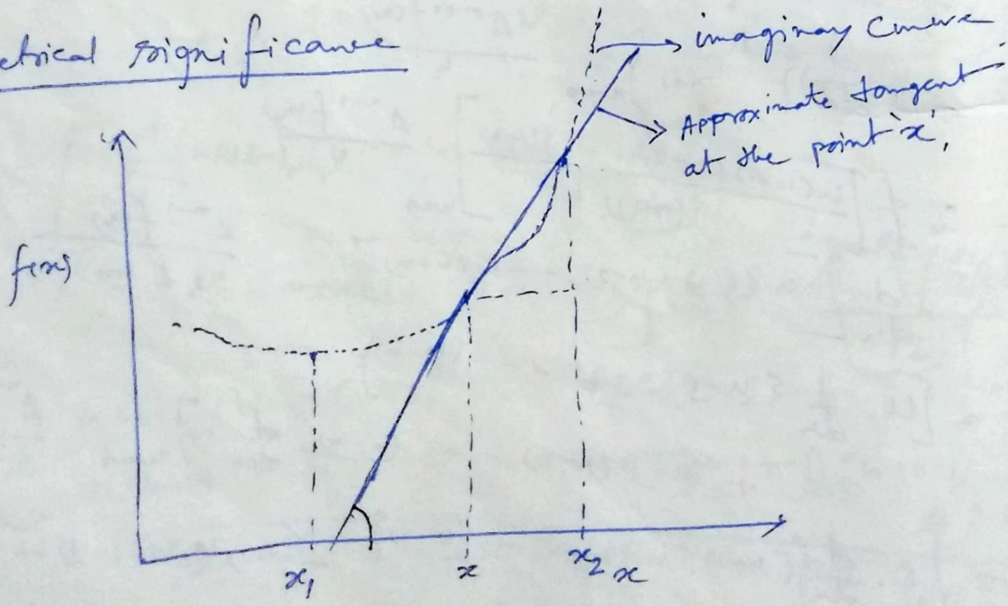
Algorithm for obtaining first derivative of a given function for a given value of x ~~=~~ x_0.

1. Obtain ~~a~~ forward difference table by using ~~newton~~ forward difference calculus. (Algorithm given previously)
2. set ~~n~~ n = no. of field's that is difference column in your forward difference table.
3. set sum = 0, $i=2$
4. Repeat step 5 ~~&~~ until $i<=n$, and $i=i+1$ in every loop.
5. ~~for~~ sum = sum + $(-1)^{i}$ · difference[1][i]/(i-1)
6. set sum = sum/h. [h is the distance between two points].
7. set $f'(x_0)$ = sum.
8. exit.

In two case we need numerical differentiation.

1. if function is known for some tabulated value then cannot ~~do~~ have the derivative of that function.
2. The function needed to ~~derivate~~ differentiate is complicated and would not easy ~~to get~~ to get ~~for~~ derivative.

Geometrical significance



Differentiation formula based on Newton's backward ~~formula~~

Interpolation formula

We know that $\{x_0, f(x_0)\}, \{x_1, f(x_1)\}, \ldots \{x_n, f(x_n)\}$, points

are known,

where $a = x_0$,

$x_1 = x_0 + h$

$x_2 = x_1 + h$

$x_n = x_{n-1} + h$

$x_n = b$.

and $x_1 = x_0 + rh$

and $x_{n-r} = x_n - rh$

Now we know the backward Interpolation formula of Newton's is $\rightarrow$

$$f(x) = f(x_n) + u \cdot \nabla y_n + \frac{u(u+1)}{2!} \nabla^2 y_n + \frac{u(u+1)(u+2)}{3!} \nabla^3 y_n + \cdots + u(u+1)(u+2) \cdots (u+n-1) \frac{\nabla^n y_n}{n!}$$

where $u = \dfrac{x - x_n}{h}$, h = length of each interval

now,

$$f(x) = f(x_n) + u \cdot \nabla y_n + \frac{1}{2}(u^2+u) \cdot \nabla^2 y_n + \frac{1}{6}(u^3 + 3u^2 + 2u) \cdot \nabla^3 y_n + \frac{1}{24}(u^4 + 6u^3 + 11u^2 + 6u) \cdot \nabla^4 y_n \cdots$$

now differentiating both side of $f(x)$ we have (with respect to u)

$$f'(x) \frac{dx}{du} = 0 + \nabla y_n + \frac{1}{2}(2u+1) \cdot \nabla^2 y_n + \frac{1}{6}(3u^2+6u+2) \cdot \nabla^3 y_n + \frac{1}{24}(4u^3+18u^2+22u+6) \cdot \nabla^4 y_n \cdots$$

$$\therefore f'(x) \cdot \frac{dx}{du} = \nabla y_n + \left(u+\frac{1}{2}\right) \cdot \nabla^2 y_n + \left(\frac{u^2}{2} + u + \frac{1}{3}\right) \nabla^3 y_n + \left(\frac{u^3}{6} + \frac{3}{4}u^2 + \frac{11}{12}u + \frac{1}{4}\right) \nabla^4 y_n + \cdots$$

now, $u = \dfrac{x - x_n}{h}$

or, $u \cdot h = (x - x_n)$

$\therefore$ differentiating both side with respect to x.

we have

$$h \cdot \frac{du}{dx} = 1 - 0 = 1$$

$$\therefore \frac{dx}{du} = h \cdot \quad \therefore \frac{du}{dx} = \frac{1}{h}$$

Therefore

$$f'(x) \cdot h = \Big[\nabla y_n + \left(u+\tfrac{1}{2}\right)\nabla^2 y_n + \left(\tfrac{u^2}{2} + u + \tfrac{1}{3}\right) \cdot \nabla^3 y_n + \left(\tfrac{u^3}{6} + \tfrac{3}{4}u^2 + \tfrac{11}{12}u + \tfrac{1}{4}\right)\nabla^4 y_4 + \cdots\Big]$$

$$\therefore f'(x) = \frac{1}{h}\Big[\nabla y_n + \left(u+\tfrac{1}{2}\right)\nabla^2 y_n + \left(\tfrac{u^2}{2} + u + \tfrac{1}{3}\right) \cdot \nabla^3 y_n + \left(\tfrac{u^3}{6} + \tfrac{3}{4}u^2 + \tfrac{11}{12}u + \tfrac{1}{4}\right)\nabla^4 y_n + \cdots\Big]$$

Formula for first derivative.

Again differentiating both side with respect to u we get.

$$\therefore f''(x) \cdot \frac{dx}{du} = \frac{1}{h}\Big[0 + (1+0)\nabla^2 y_n + (u+1)\nabla^3 y_n + \left(\tfrac{u^2}{2} + \tfrac{3}{2}u + \tfrac{11}{12}\right) \cdot \nabla^4 y_n + \cdots\Big]$$

$$\therefore f''(x) \cdot h = \frac{1}{h}\Big[\nabla^2 y_n + (u+1)\nabla^3 y_n + \left(\tfrac{u^2}{2} + \tfrac{3}{2}u + \tfrac{11}{12}\right)\nabla^4 y_n + \cdots\Big]$$

$$\therefore f''(x) = \frac{1}{h^2}\Big[\nabla^2 y_n + (u+1)\nabla^3 y_n + \left(\tfrac{u^2}{2} + \tfrac{3}{2}u + \tfrac{11}{12}\right)\nabla^4 y_n + \cdots\Big]$$

Again differentiating both side with respect to u we get,

$$f'''(x) \cdot \frac{dx}{du} = \frac{1}{h^2}\Big[0 + (1+0)\nabla^3 y_n + \left(u+\tfrac{3}{2}\right)\nabla^4 y_n + \cdots\Big]$$

$$\text{or, } f'''(x) = \frac{du}{dx} \cdot \frac{1}{h^2}\Big[\nabla^3 y_n + \left(u+\tfrac{3}{2}\right) \cdot \nabla^4 y_n + \cdots\Big]$$

$$\therefore f'''(x) = \frac{1}{h}\cdot\frac{1}{h^2}\left[\nabla^3 y_n + \left(u+\frac{3}{2}\right)\nabla^4 y_n + \cdots\right]$$

$$\therefore f'''(x) = \frac{1}{h^3}\left[\nabla^3 y_n + \left(u+\frac{3}{2}\right)\nabla^4 y_n + \left(\frac{u^2}{2} + 2u + \frac{7}{4}\right)\nabla^5 y_n + \cdots\right]$$

now if $x = x_n$ then

$$u = \frac{x - x_n}{h} = \frac{x_n - x_n}{h} = \frac{0}{h} = 0.$$

$\therefore$ if $x = x_n$, then $u = 0$.

$$f'''(x_n) = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \frac{7}{4}\nabla^5 y_n + \cdots\right]$$

$$f''(x_n) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right]$$

$$f'(x_n) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \cdots\right]$$

like this way we can obtain $f^{n+1}(x)$, or limit to our desire.

Now estimating error in differentiation formula based on newton's backward interpolation formula.

We know that error committed in newtons backward interpolation formula is

$$R_{n+1}(x) = u(u+1)(u+2)\cdots(u+n-1)(u+n)\,h^{n+1}\,\frac{f^{n+1}(\xi)}{(n+1)!}$$

where $u = \dfrac{x - x_n}{h}$ $\therefore \dfrac{dx}{du} = h.$

now differentiating it with respect to 'u' we get.

$$R'_{n+1}(x)\cdot\frac{dx}{du} = \frac{h^{n+1}}{(n+1)!}\frac{d}{du}\left[u(u+1)(u+2)\cdots(u+n)\cdot f^{n+1}(x)\right]$$

$$\Rightarrow R'_{n+1}(x)\cdot h = \frac{h^{n+1}}{(n+1)!}\left[f^{n+2}(x)\cdot\left(\frac{dx}{du}\right)\cdot u(u+1)(u+2)\cdots(u+n) + f^{n+1}(x)\cdot\frac{d}{du}\{u(u+1)\cdots(u+n)\}\right]$$

$$\therefore R'_{n+1}(x) = \frac{h^n}{(n+1)!}\Big[h\cdot f^{n+2}(x)\cdot u(u+1)(u+2)\cdots(u+n)$$

$$+ f^{n+1}(x)\cdot \frac{d}{du}\{u(u+1)\cdots(u+n)\}\Big]$$

$$= \frac{h^{n+1}}{(n+1)!}\cdot f^{n+2}(x)\cdot u(u+1)\cdot(u+2)\cdots(u+n)$$

$$+ \frac{h^n f^{n+1}(x)}{(n+1)!}\cdot\frac{d}{du}\{u(u+1)\cdots(u+n)\}$$

nono $u(u+1)(u+2)\cdots(u+n)$

$$= \frac{(u+n)(u+n-1)(u+n-2)\cdots(u+2)(u+1)u(u-1)(u-2)\cdots 3\,2}{(u-1)(u-2)\cdots\cdots 3\cdot2\cdot1}$$

$$= \frac{(u+n)!}{(u-1)!}$$

$$= \frac{(u+n)!}{\{(u+n)-(n+1)\}!}$$

now. $$R'_{n+1}(x) = \frac{h^{n+1}}{\cancel{(n+1)!}}\cdot f^{n+2}(x_0)\cdot\frac{(u+n)!}{\{(u+n)-(n+1)\}!\cdot(n+1)!}$$

$$+ \frac{h^n f^{n+1}(x)}{}\cdot\frac{d}{du}\left\{\frac{(u+n)!}{\{(u+n)-(n+1)\}!\cdot(n+1)!}\right\}$$

$$= h^{n+1} f^{n+2}(x)\cdot {}^{u+n}C_{n+1}$$

$$+ h^n f^{n+1}(x)\cdot\frac{d}{du}\left({}^{u+n}C_{n+1}\right)$$

$$= \left({}^{u+n}C_{n+1}\right)\frac{\Delta^{n+2} f(x_0)}{(n+2)\cdot h}$$

$$+ \frac{d}{du}\left({}^{u+n}C_{n+1}\right)\cdot\frac{\Delta^{n+1} f(x_0)}{h}$$

now. if $x = x_n$, i.e $u = 0$. then

$$R'_{n+1}(x_n) \cdot \frac{dx}{du} = \frac{h^{n+1}}{(n+1)!}\left[f^{n+2}(x) \cdot \left(\frac{dx}{du}\right) \cdot u(u+1)(u+2)\cdots(u+n) + f^{n+1}(x) \cdot \frac{d}{du}\{u(u+1)\cdots(u+n)\}\right]$$

if $u = 0$

$$= \frac{h^{n+1}}{(n+1)!}\left[f^{n+2}(x)\left(\frac{dx}{du}\right) \cdot 0 \cdot (0+1)(0+2)\cdots(0+n) + f^{n+1}(x) \cdot \frac{d}{du}\{u(u+1)\cdots(u+n)\}_{u=0}\right]$$

$$= \frac{h^{n+1}}{(n+1)!}\left[0 + f^{n+1}(x) \cdot \frac{d}{du}\{u(u+1)\cdots(u+n)\}_{u=0}\right]$$

$$= \frac{h^{n+1}}{(n+1)!} \cdot f^{n+1}(x) \cdot \frac{d}{du}\{u(u+1)\cdots(u+n)\}_{u=0}$$

$$\therefore R'_{n+1}(x_n)\frac{dx}{du} = \frac{h^{n+1} f^{n+1}(x)}{(n+1)!} \cdot \frac{d}{du}\{u(u+1)\cdots(u+n)\}_{u=0}$$

$$= \frac{\Delta^{n+1} f(x_0)}{(n+1)!} \cdot \left[(u+1)(u+2)\cdots(u+n) \cdot \frac{d}{du}(u) + u \cdot \frac{d}{du}\{(u+1)(u+2)\cdots(u+n)\}\right]_{u=0}$$

$$= \frac{\Delta^{n+1} f(x_0)}{(n+1)!} \cdot \left[(0+1)(0+2)\cdots(0+n) \cdot 1 + 0 \cdot \frac{d}{du}\{(u+1)(u+2)\cdots(u+n)\}\right]$$

$$= \frac{\Delta^{n+1} f(x_0)}{(n+1)!} \cdot [n! + 0]$$

$$= \frac{\Delta^{n+1} f(x_0)}{(n+1)!} \cdot n!$$

$$= \frac{\Delta^{n+1} f(x_0)}{(n+1)}$$

$$\therefore R'_{n+1}(x_n) \cdot h = \frac{\Delta^{n+1} f(x_0)}{(n+1)}$$

$$\therefore R'_{n+1}(x_n) = \frac{\Delta^{n+1} f(x_0)}{(n+1) \cdot h}$$

Calculation [illegible] → notification

Algorithm for obtaining first derivative of given function for a given value of $x = x_n$ [use newtons backward interpolation formula].

1. Obtain backward difference table by using backward difference calculus and algorithm for it given previously.
2. Set n = no of field's that is difference columns in your backward difference table.
3. set sum = 0, i = 1,
4. Repeat step 5 untill $i <= n$, and $i = i+1$ in every loop
5. set sum = sum + difference[n][i]/i
6. set sum = sum/h [where h is the distance between two points].
7. set $f'(x_n)$ = sum
8. exit.

Q. The function f(x) is given below

x :	0	5	10	15	20
f(x) :	1.5708	1.5738	1.5828	1.5981	1.6200

Compute the first ~~and second~~ order derivatives of f(x) at x=0, 20, 3. Use forward difference table.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1.5708	0.0030	0.0060	0.0003
5	1.5738	0.0090	0.0063	0.0003
10	1.5828	0.0153	0.0066	
15	1.5981	0.0219		
20	1.6200			

at $x = 0$,

The forward interpolation formula based derivative of $f(x)$ is,

$$f'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \cdots\right]$$

where $u = \frac{x-x_0}{h}$. where $x_0 = 0$, $x = 0$, $\therefore u = 0$.

$$\therefore f'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2.0-1}{2}.\Delta^2 y_0 + \frac{3.0^2-6.0+2}{6}.\Delta^3 y_0 + \cdots\right]$$

$$= \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{2}{6}.\Delta^3 y_0 + \cdots\right] \text{ [putting zero]}$$

putting zero

$$f'(0) = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} + \cdots\right]$$

here $h = (5-0) = 5$.

$\Delta y_0 =$ ~~1.1708~~ 0.0030, $\Delta^2 y_0 = .0060$, $\Delta^3 y_0 = 0.0003$

$$\therefore f'(0) = \frac{1}{5}\left[0.0030 - \frac{.0060}{2} + \frac{0.0003}{3}\right]$$

$$= \frac{1}{5}\left[\cancel{0.0030} - \cancel{.0030} + 0.0001\right] = \frac{0.0001}{5}$$

$$\therefore f'(0) = 0.00002.$$

at $x = 3$, $u = \frac{3-0}{5} = 3/5$

$$\therefore f'(3) = \frac{1}{5}\left[\Delta y_0 + \frac{2.3/5-1}{2}.\Delta^2 y_0 + \frac{3.\frac{9}{25} - 6.\frac{3}{5} + 2}{6}\Delta^3 y_0 + \cdots\right]$$

$$= \frac{1}{5}\left[\Delta y_0 + \frac{1}{10}\Delta^2 y_0 + -\frac{3}{25.62}.\Delta^3 y_0 + \cdots\right]$$

$$= \frac{1}{5}\left[\Delta y_0 + \frac{1}{10}.\Delta^2 y_0 - \frac{1}{50}\Delta^3 y_0 + \cdots\right]$$

$$= \frac{1}{5}\left[0.0030 + \frac{1}{10}.0.0060 - \frac{1}{50}.0.000300\right]$$

$$= \frac{1}{5}\left[0.0030 + 0.0006 - 0.000006\right]$$

$$= \frac{1}{5}\left[0.0036 - 0.000006\right]$$

$$= \frac{1}{5}.0.003594 = 0.000708$$

next. $f'(x)$ at $x=20$,

$u = \frac{20-0}{5} = 4$, $h=5$, $x_0 = 0$

$\therefore f'(20) = \frac{1}{5}\left[\Delta y_0 + \frac{2u-1}{2}\cdot\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \cdots\right]$

$= \frac{1}{5}\left[\Delta y_0 + \frac{2\cdot4-1}{2}\cdot\Delta^2 y_0 + \frac{3\cdot4^2-6\cdot4+2}{6}\cdot\Delta^3 y_0 + \cdots\right]$

$= \frac{1}{5}\left[\Delta y_0 + \frac{7}{2}\cdot\Delta^2 y_0 + \frac{\cancel{26}^{13}}{\cancel{6}_3}\Delta^3 y_0\right]$

$= \frac{1}{5}\left[\Delta y_0 + \frac{7}{2}\cdot\Delta^2 y_0 + \frac{13}{3}\Delta^3 y_0\right]$

$= \frac{1}{5}\left[0.0030 + \frac{7}{\cancel{2}}\cdot 0.00\cancel{60}^{30} + \frac{13}{\cancel{3}}\cdot 0.000\cancel{3}^{1}\right]$

$= \frac{1}{5}\left[0.0030 + 0.0210 + 0.0073\right]$

$= \frac{1}{5}\left[0.0240 + 0.0013\right]$

$= \frac{1}{5}\cdot 0.0253 = 0.00506$.

$\therefore f'(20) = 0.00506$.

Q. Do same sum with newtons backward interpolation formula.

Ans. The newtons backward ~~use~~ interpolation formula

is, $f'(x) = \frac{1}{h}\left[\nabla y_n + \left(u+\frac{1}{2}\right)\cdot\nabla^2 y_n + \left(\frac{u^2}{2}+u+\frac{1}{3}\right)\nabla^3 y_n + \left(\frac{u^3}{6}+\frac{3}{4}u^2+\frac{11}{12}u+\frac{1}{4}\right)\nabla^4 y_n + \cdots\right]$

where $u = \frac{x-x_n}{h}$

here $h=5$, $x_n = 20$

$\therefore u = \frac{x-20}{5} = \frac{x-20}{5}$

now f at $x=20$.

$u = \frac{20-20}{5} = \frac{0}{5} = 0$

$\therefore f'(20) = \frac{1}{5}\left[\nabla y_n + \left(0+\frac{1}{2}\right)\cdot\nabla^2 y_n + \left(\frac{0^2}{2}+0+\frac{1}{3}\right)\cdot\nabla^3 y_n + \dots\right]$

$= \frac{1}{5}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \dots\right]$

$= \frac{1}{5}\left[0.0219 + \frac{0.0066}{2} + \frac{0.0003}{3}\right]$

$= \frac{1}{5}\left[0.0219 + 0.0033 + 0.0001\right]$

$= \frac{1}{5}\left[0.0220 + 0.0033\right]$

$= \frac{1}{5}\left[0.0253\right] = 0.00506$

$\therefore f'(20) = 0.00506.$

at $x = 0$. $u = \frac{0-20}{5} = -4.$

$\therefore f'(0) = \frac{1}{5}\left[\nabla y_n + \left(-4+\frac{1}{2}\right)\cdot\nabla^2 y_n + \left(\frac{16}{2}-4+\frac{1}{3}\right)\cdot\nabla^3 y_n + \dots\right]$

$= \frac{1}{5}\left[\nabla y_n - \frac{7}{2}\cdot\nabla^2 y_n + \frac{13}{3}\cdot\nabla^3 y_n\right]$

$= \frac{1}{5}\left[0.0219 - \frac{7}{\cancel{2}}\cdot 0.00\cancel{66}^{33} + \frac{13}{\cancel{3}}\cdot 0.000\cancel{3}^{1}\right]$

$= \frac{1}{5}\left[0.0219 - 0.0231 + 0.0013\right]$

$= \frac{1}{5}\left[0.0232 - 0.0231\right] = 0.0001/5$

$\therefore f'(0) = 0.00002.$

at, $x = 3$,

$u = \frac{3-20}{5} = -\frac{17}{5}$

$\therefore f'(3) = \frac{1}{5}\left[\nabla y_n + \left(-\frac{17}{5}+\frac{1}{2}\right)\cdot\nabla^2 y_n + \left(\frac{289}{50}-\frac{17}{5}+\frac{1}{3}\right)\cdot\nabla^3 y_n + \dots\right]$

$\therefore f'(3) = \frac{1}{5}\left[\nabla y_n - \frac{29}{10}\cdot\nabla^2 y_n + \left(\frac{129}{50}+\frac{1}{3}\right)\nabla^3 y_n\right]$

$= \frac{1}{5}\left[\nabla y_n - \frac{29}{10}\cdot\nabla^2 y_n + \frac{437}{150}\nabla^3 y_n\right]$

$$\therefore f'(3) = \frac{1}{5}\left[0.0219 - \frac{29}{10} \cdot 0.0066 + \frac{437}{150} \cdot 0.0003\right]$$

$$= \frac{1}{5}\left[0.0219 - 29 \cdot 0.00066 + \frac{874}{\cancel{300}\ 100} \cdot 0.000\overset{1}{\cancel{3}}\right]$$

$$= \frac{1}{5}\left[0.0219 - 0.01914 + 8.74 \cdot 0.0001\right]$$

$$= \frac{1}{5}\left[0.0219 - 0.01914 + 0.000874\right]$$

$$= \frac{1}{5} \cdot 0.003634$$

$$= \underline{.0007268}$$

0.0219
0.000874
0.022774
0.019140
5) 0.003634 (.0007268
35
13
10
34
30
40
40
0

Problems