(13). **Def^n:-** A formula is said to be closed if there are no free occurances of any variable in it.

Eg: $(\exists x)(\forall y)(P(x,y) \wedge Q(x))$

Here x, y both are under quantifiers scope. and therefore the ~~for~~ formula is closed.

But $\forall_y P(x,y)$, Here x is not under scope of $(\forall y)$, therefore the formula is not closed.

Def^n:- If α is a formula then $\forall(\alpha)$ denotes the universal closure of α which is a closed formula.

Def^n:- If α is a formula then $\exists(\alpha)$ denotes the existential closure of α which is a closed formula.

Eg: Consider a formula α : $P(x,y) \wedge Q(x)$

$\therefore$ Universal clousure of α is $\forall(\alpha) : (\forall x)(\forall y)(P(x,y) \wedge Q(x))$

" Existential closure of α is $\exists(\alpha) : (\exists x)(\exists y)(P(x,y) \wedge Q(x))$

First order Predicale logic:-

An addition of inference rules with First order predicate calculus is called First order predicate logic.

Def^n: An interpretation of a formula α in FOPL consists of
1) A non empty domain D
2) An assignment of values to each constant
3) function symbols
4) Predicate symbols occuring in α.

~~Deto~~ Denoted by I.

Defⁿ:- Assume α and β are formulae and I is an interpretation over any domain D. Then following holds.

- $I[\alpha \wedge \beta] = I[\alpha] \wedge I[\beta]$
- $I[\alpha \vee \beta] = I[\alpha] \vee I[\beta]$
- $I[\alpha \to \beta] = I[\alpha] \to I[\beta]$
- $I[\sim\alpha] = \sim I[\alpha]$

Defⁿ:- Any interpretation I and a formula using $(\forall x)$ and $(\exists x)$ the following holds are true.

$$I[(\forall x)\, P(x)] = T \text{ iff } I[P(x)] = T,\ \forall x \in D$$
$$= F \text{ otherwise}$$

$$I[(\exists x)\, P(x)] = T \text{ iff } \exists x \in D \text{ such that } I[P(x)] = T$$
$$= F \text{ otherwise.}$$

Eg: Consider α and β to be two formulae and the following interpretations:

$\alpha : (\forall x)\, P(x)$

$\beta : (\exists x) \sim P(x)$.

I

$D = \{1, 2\}$
$I[P(1)] = T,\ I[P(2)] = F$

Ans. $\therefore I[\alpha] = I[(\forall x)\, P(x)]$

$$= I[P(1) \wedge P(2)]$$
$$= I[P(1)] \wedge I[P(2)]$$
$$= T \wedge F$$
$$= F$$

$I[\beta] = I[(\exists x) \sim P(x)]$

Here we see that $\exists 2 \in D$ such that
$I[P(x)] = F$ therefore

(ii) $I[\beta] = I[\sim P(2)]$
$= \sim I[P(2)]$
$= \sim F$
$= T.$

Therefore α and β are evaluated false and true respectively under interpretation I.

Eg:- Let $\alpha: (\forall x)(\exists y)\, P(x,y)$ be a formula. Evaluate it under the following interpretation I.

I

$D = \{1, 2\}$	
$I[P(1,1)] = F$,	$I[P(2,1)] = T$
$I[P(1,2)] = T$,	$I[P(2,2)] = F$

Ans.

if $x = 1$, then $\exists 2 \in D$ such that $I[P(1,2)] = T$

if $x = 2$ " $\exists 1 \in D$ such that $I[P(2,1)] = T$.

Therefore $I[\alpha] = I[(\forall x)(\exists y)\, P(x,y)] = T$

$\therefore$ Therefore α is true under above interpretation

Eg: Consider a formula $\alpha: (\forall x)(P(x) \rightarrow Q(f(x), c))$

Find the truth value under the following interpretation

I

$D = \{1, 2\}$

$c = 1$, is a constant in domain D

$I[f(1)] = 2$, $I[f(2)] = 1$

$I[P(1)] = F$, $I[P(2)] = T$

$I[Q(1,1)] = T$, $I[Q(1,2)] = T$

$I[Q(2,1)] = F$ $I[Q(2,2)] = T$

Ans: Now the formula is

$$\alpha : (\forall x)\left(P(x) \rightarrow Q(f(x), c)\right)$$

then $I[\alpha] = I\left[(\forall x)\left(P(x) \rightarrow Q(f(x), c)\right)\right]$

$$= I\left[P(1) \rightarrow Q(f(1), 1)\right] \wedge I\left[P(2) \rightarrow Q(f(2), 1)\right]$$

[As $c = 1$, $f(1) = 2$, $f(2) = 1$ and $D = \{1, 2\}$]

$$= \left\{I[P(1)] \rightarrow I[Q(f(1), 1)]\right\}$$
$$\wedge \left\{I[P(2)] \rightarrow I[Q(f(2), 1)]\right\}$$

$$= \left\{F \rightarrow I[Q(f(1), 1)]\right\} \wedge \left\{T \rightarrow I[Q(f(2), 1)]\right\}$$

$$\therefore I[\alpha] = T \wedge \left\{T \rightarrow I[Q(f(2), 1)]\right\}$$
$$= T \wedge \left\{T \rightarrow I\left[Q(I[f(2)], 1)\right]\right\}$$
$$= T \wedge \left\{T \rightarrow I[Q(1, 1)]\right\}$$
$$= T \wedge \{T \rightarrow T\}$$
$$= T \wedge T$$
$$= T$$

$\therefore$ Therefore α is true under above interpretation I.

Eg: Find the truth values of α and β under the following interpretation I.

α: Less $(f(c), f(f(c)))$

β: Less $(f(f(c)), f(c))$

I

D = Set of Natural no.
$I[c] = 0$, c = constant
$I[f(t)] = I[t] + 1$
$I[\text{Less}(t_1, t_2)] = T$ if $I[t_1] < I[t_2]$

(14)

Ans $I[\alpha] = I[less(f(c), f(f(c)))]$

now $I[f(c)] < I[f(f(c))]$

$= \{I[c]+1\} < I[f(c)]+1$

$= (0+1) < I[c]+1+1$

$= 1 < 0+2$

$= 1 < 2$

$= T$

$\therefore I[\alpha] = I[less(f(c), f(f(c)))] = I[f(c)] < I[f(f(c))]$

$= T$

$\therefore I[\alpha] = T.$

now, $I[\beta] = I[less(f(f(c)), f(c))]$

now. $I[f(f(c)), f(c)]$

$= I[f(f(c))] < I[f(c)]$

$= I[f(c)]+1 < I[c]+1$

$= I[c]+1+1 < I[c]+1$

$= 0+2 < 0+1$

$= 2 < 1$

$= F$

$\therefore I[\beta] = F$

So from this we can conclude that $I[\alpha] = T$, $I[\beta] = F$.

Defn:- A formula α is said to be consistent (satisfiable) if and only if there exists an interpretation I such that $I[\alpha] = T$, ~~Alter~~ In other words I is a model of α or I satisfies α.

Defn:- A formula α is said to be inconsistent if and only if $\exists$ no interpretation that satisfies α.

Def: A formula α is valid if and only if for every interpretation I, $I[\alpha] = T$.

Def: A formula α is a logical consequence of a set of formula $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ if and only if for every interpretation I, if $I[\alpha_1 \wedge \alpha_2 \ldots \alpha_n] = T$ then $I[\alpha] = T$.

Inference rules in Predicate logic

Modus Ponen Rule:-

Lemma 1: If $\alpha : (\forall x)(P(x) \to Q(x))$, $\beta : P(c)$ are two formulae, then $Q(c)$ is a logical consequence of α and β where 'c' is a constant.

Proof: Consider and interpretation I, over any domain D

$I[\alpha \wedge \beta] = T$, then we have to verify that $I[Q(c)] = T$

Let, $I[Q(c)] = F$

now, $I[\alpha \wedge \beta] = T$

$\Rightarrow I[\alpha] \wedge I[\beta] = T$

$I[\alpha] = T, \quad I[\beta] = T$

$\therefore I[(\forall x)(P(x) \to Q(x))] = T$ --- ①

$I[P(c)] = T$. --- ②

Also now $I[Q(c)] = F$

and $I[P(c) \to Q(c)]$

$= I[P(c)] \to I[Q(c)]$

$= T \to F$ --- ③

$= F$

But from (1) we have $I[(\forall x)(P(x) \to Q(x))] = T$ given, but from 3, we have $\exists c \in D$, such that $I[P(c) \to Q(c)] = F$.

* But it is impossible, therefore whatever we have assumed is wrong.

$\therefore I[\varphi(c)] = T.$

$\therefore$ therefore iff $I[\alpha \wedge \beta] = T$ then $I[\varphi(c)] = T$.

therefore $\varphi(c)$ is the logical consequence of $\alpha \wedge \beta$.

Modus Tollens Rule:

Lemma 2. Iff $\alpha: (\forall x)(P(x) \rightarrow \varphi(x))$ and $\beta: \sim\varphi(c)$ are two formulae, then $\sim P(c)$ is the logical consequence of α and β where 'c' is a constant.

Proof: Consider an interpretation I over some domain D, $I[\alpha \wedge \beta] = T$.

then according to the problem we have to proof that $I[\sim P(c)] = T$. i.e $I[P(c)] = F$

Now let us assume $I[P(c)] = T$.

now, $I[\alpha \wedge \beta] = T$

or $I[\alpha] \wedge I[\beta] = T$.

or $I[\alpha] = T$

$I[\beta] = T$.

as, $I[\beta] = T$, $\therefore I[\sim\varphi(c)] = T$

$\therefore I[\varphi(c)] = \sim T = F$

$I[\varphi(c)] = F$. — (1)

now $I[\alpha] = T$.

therefore $I[(\forall x)(P(x) \rightarrow \varphi(x))] = T$.

$\therefore I[P(c) \rightarrow \varphi(c)] = T$. [for some $c \in D$]

or, $I[P(c)] \rightarrow I[\varphi(c)] = T$

or, $I[P(c)] \rightarrow F = T$. — (2)

From 2, we can say it is only possible when $I[P(c)] = F$

Therefore whatever we assumed is false

$\therefore I[P(c)] \neq T.$

$\therefore I[P(c)] = F.$

$\therefore I[\sim P(c)] = T.$

Therefore ~~for each~~ we can conclude that

$I[\alpha \wedge \beta] = T$ when $I[\sim P(c)] = T$

$\therefore$ $\sim P(c)$ is the logical consequence of $\alpha \wedge \beta$.

Eg: Show that δ is a logical consequence of α & β

$\alpha: (\forall x)(P(x) \rightarrow \sim Q(x))$

$\beta: (\exists x)(Q(x) \wedge R(x)).$

$\delta: (\exists x)(R(x) \wedge \sim P(x))$

Ans.: let I be any interpretation over some domain D.

$\therefore$ from the problem we have to show that

if $I[\alpha \wedge \beta] = T$, then we have to proof that $I[\delta] = T$

Now. $I[\alpha \wedge \beta] = T$

$\therefore I[\alpha] = T$ and $I[\beta] = T$

now. $I[(\forall x)\ P(x) \rightarrow \sim Q(x)] = T$ — (1)

and $I[(\exists x)(Q(x) \wedge R(x))] = T$ — (2)

let there exists an element $c \in D$, from (2)

therefore $I[Q(c) \wedge R(c)] = T$ — (3)

$I[Q(c)] = T$ — (4)

$\therefore I[R(c)] = T$

now, from (3) we can write $I[\sim Q(c)] = F.$ — (5)

now, from 1 we get

$I[P(c) \rightarrow \sim Q(c)] = T$ as it is true for all domain

$\therefore I[P(c)] \rightarrow I[\sim Q(c)] = T$

(5) $\sim I[P(c)] \rightarrow F = T$

Therefore $I[P(c)] = F$

$\therefore I[\sim P(c)] = T$ (7)

now $I[\delta] = I[(\exists x)(R(x) \wedge \sim P(x))]$

let $x = c$, $I[\delta] = I[R(c) \wedge \sim P(c)]$

$= I[R(c)] \wedge I[\sim P(c)]$

$= T \wedge T$ from 4, 7

$= T$

$\therefore$ It is proved that if $I[\alpha \wedge \beta] = T$, then $I[\delta] = T$

$\therefore$ δ is the logical consequence of $\alpha \wedge \beta$.

Prenex normal form:

A closed formula α of FOPL is said to be in Prenex Normal form (PNF) if and only if α is represented as $(q_1 x_1)(q_2 x_2) \cdots (q_n x_n)(M)$ where q_k is either $\forall$ or $\exists$ quantifier, M is a formula free of quantifiers and $(1 \le k \le n)$.

for example $(\forall x)(\exists y)(P(x) \wedge Q(x, y))$ are in prenex normal form. The list of quantifiers for a formula is called prefix of that formula and (M) is called matrix of that formula.

Therefore $\alpha: (\forall x)(\exists y)[P(x) \wedge Q(x, y))$

where $(\forall x)(\exists y)$ is prefix

and $P(x) \wedge Q(x, y)$ is matrix

and the formula α is in PNF.

But, $(\forall x)(P(x) \rightarrow (\exists y) Q(x, y))$ is not in prenex normal form.